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Academic dissertation
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Faculty of Science
University of Helsinki

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Abstract

The problem of recovering information from measurement data has already been studied for a long time. In the beginning, the methods were mostly empirical, but already towards the end of the sixties Backus and Gilbert started the development of mathematical methods for the interpretation of geophysical data.

The problem of recovering information about a physical phenomenon from measurement data is an inverse problem. Throughout this work, the statistical inversion method is used to obtain a solution.

Assuming that the measurement vector is a realization of fractional Brownian motion, the goal is to retrieve the amplitude and the Hurst parameter. We prove that under some conditions, the solution of the discretized problem coincides with the solution of the corresponding continuous problem as the number of observations tends to infinity.

The measurement data is usually noisy, and we assume the data to be the sum of two vectors: the trend and the noise. Both vectors are supposed to be realizations of fractional Brownian motions, and the goal is to retrieve their parameters using the statistical inversion method. We prove a partial uniqueness of the solution. Moreover, with the support of numerical simulations, we show that in certain cases the solution is reliable and the reconstruction of the trend vector is quite accurate.

Ai miei tre porcellini

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Contents

Introduction

An inverse problem framework consists of a physical model M, a set of model parameters (not all directly measurable) describing M and a set of observable parameters whose values depend on the values of the model parameters. Solving the *forward problem* means calculating the values of the observable parameters given the values of the model parameters. On the other hand, solving the *inverse problem* means inferring the values of the model parameters given the values of the observable parameters.

In this work we consider the problem of recovering information from geophysical data. Like most of the natural phenomena, geophysical data are measured directly and indirectly. For example, the temperature of the air at different altitudes can be measured directly. On the other hand, the ozone density in the stratosphere cannot be measured directly. However, it is possible to retrieve it, since it is known from physics that the ozone density is a function of pressure (see [5]).

Both measurement vectors, the air temperature and ozone density, are affected by noise due to the climatic conditions and the measuring instrument accuracies and electronics. In this example, the inverse problem would be to retrieve the trend by eliminating the noise from the measurement vector.

For a long time, the methods for extracting information from data were mostly empirical. Backus and Gilbert made a systematic study of the mathematical structure at the basis of the inverse problems and started the development of methods for the interpretation of geophysical data (see, e.g. [1], [2]).

To recover the solution of the inverse problem in this work, we use the statistical inversion method. The statistical inversion method is well known among scientists dealing with measurements of natural phenomena (see [20, 21]). The method is theoretically simple and gives surprisingly good results. The characteristic of this method is to think that the measurement vector is actually a realization of a stochastic process. The statistical inversion method assumes that all the information about the measurement vector is contained in this distribution. The nature of the inversion problem is further described by an appropriate a priori, *prior*, probability distribution for the solution.

In this work, we approach the problem from a mathematical point of view. Usually measurements are noisy. One of the most difficult problems in recovering the solution is noise estimation. Usually the phenomenon we want to recover has a non-stationary trend. The measurement data is then the sum of the trend and noise. Here, we assume that both the trend and the error are realizations of fractional Brownian motions with different parameters. We suppose that we have no a priori knowledge about the underlying parameters of the two fractional Brownian motions and use the stochastic inversion method with a constant prior distribution.

This work has two main goals. The first one is to prove that the solution of the discrete problem associated with a continuous problem coincides with the solution of the original continuous problem as the number of observations tends to infinity. The second goal of this work is to prove the uniqueness of the solution of the problem to recover the two fractional Brownian motions from their sum.

This will be done in several steps. In the first chapter, we consider a realization of Brownian motions and calculate the a posteriori, *posterior*, distribution of the amplitude parameter. We prove that the posterior distribution concentrates on the correct value of the parameter when the number of measurements tends to infinity.

This answers a question always present when recovering information from geophysical measurements. The models studied are usually continuous in time, but the measurements are performed on a discrete set of time variables. The question is whether this affects the reliability of the solution. We prove that the solution of the discretized problem tends to the solution of the continuous problem when the number of measurements tends to infinity.

In the second chapter, we generalize the result of the first chapter to the case of a realization of fractional Brownian motion with a fixed Hurst parameter. The main difficulty in the analysis is due to the complicated form of the inverse of the covariance matrix, here called the *inverse covariance matrix*, of the stochastic process. In the proof, we have to make some assumptions based on numerical results, and consider a simplified version of the problem. In particular, we consider an approximation of the inverse of the exact covariance matrix, here called the *approximate inverse covariance matrix*, in order to perform the analytic calculation. Due to a technical assumption, we also need $\frac{1}{2} < H < 1$.

When estimating the Hurst parameter of a fractional Brownian motion with a fixed amplitude parameter, we prove that the statistical inversion method gives the correct solution for the estimator when the number of measurements is large enough. Also in this case, since we use the approximate inverse of the covariance matrix, we need the restriction $\frac{1}{2} < H < 1$.

In the third chapter, we consider the sum of two fractional Brownian motions. We assume that the measurement vector is a stochastic process consisting of the sum of two realizations of fractional Brownian motion. One realization represents the trend and the other one represents the noise. We prove the partial uniqueness of the solution obtained using the statistical inversion method.

In the fourth chapter, we present simulation results. We generate two realizations of fractional Brownian motion. Taking the sum of these two realizations, we obtain a measurement vector. From the posterior distribution, we retrieve the underlying parameters and reconstruct the two realizations. The simulations show that it is possible to retrieve the underlying parameters in the case $0 < H < \frac{1}{2}$. Moreover, in the case of the sum of two fractional Brownian motions, the simulations support the analytic result and show that it is only possible to retrieve the smaller of the two Hurst parameters.

In the fifth chapter, we conclude that the statistical inversion method gives satisfactory results when retrieving the trend from measurements where both the trend and the noise are assumed to be realizations of fractional Brownian motions with different underlying parameters.

Chapter 1

Retrieving the amplitude parameter of Brownian motions

We start by some concepts from probability theory that will be used below. Let $(\Omega, \mathcal{B}, P)$ denote a given complete probability space and $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra of $\mathbb{R}^n$.

Definition 1.1. A $\mathcal{B}$-measurable function $X : \Omega \to \mathbb{R}^n$ is called a **random variable**. Every random variable X induces a probability measure μ_X on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, defined by

$$\mu_X(B) = P(X^{-1}(B)). \tag{1.2}$$

The measure μ_X is called the **distribution** of X. A **stochastic process** is a collection of random variables $\{X(t), t \in I\}$ defined on a probability space $(\Omega, \mathcal{B}, P)$ with values on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

The index t represents the time (i.e. I is a subset of the real line). This means that X is a real-valued function $X(t, \omega)$ on $I \times \Omega$ which is a $\mathcal{B}$-measurable function on Ω for each $t \in I$. In this paper, we will occasionally use the notation $X(t)$ to denote the random variable $X(t, \cdot)$.

Definition 1.3. A stochastic process $\{X(t), t \in I\}$ defined on a probability space $(\Omega, \mathcal{B}, P)$ is called an **additive** process (or a process with independent increments) if for any $\{t_1, t_2, ..., t_n\} \in I, t_1 < t_2 < ... < t_n$, the system of random variables $\{X(t_{i+1}, \cdot) - X(t_i, \cdot), i = 1, 2, ..., n-1\}$ is independent.

The definition and basic properties of Brownian motions, denoted as BM from now on, follow below.

Definition 1.4. A **Brownian motion** is an additive random process B on a probability space $(\Omega, \mathcal{B}, P)$ and an interval $I \subset \mathbb{R}$ such that:

1. for fixed $\omega \in \Omega$, $B(t,\omega)$ is a continuous function with respect to t, and with probability 1 we have $B(0,\omega) = 0$ (i.e. the process starts from the origin).
2. $\forall t \geq 0$ and $h > 0$ we have

$$B(t+h,\omega) - B(t,\omega) \sim \eta(0,h)$$

where $\eta(0,h)$ is the normal distribution with mean 0 and variance h. Thus

$$P(B(t+h,\omega) - B(t,\omega) \leq x) = (2\pi h)^{-1/2} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2h}\right) du.$$

Remark 1.5. Since the distribution of $B(t+h,\omega) - B(t,\omega)$ is independent of t (i.e. B has stationary increments), assuming that $t = 0$ above, it is easy to see that $B(t,\omega) \sim \eta(0,t) \quad \forall t \in I$.

The above definition is easily extended from $\mathbb{R}$ to $\mathbb{R}^n$:

Definition 1.6. 1. Stochastic processes X_1 and X_2 defined on the probability space $(\Omega, \mathcal{B}, P)$ with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are **stochastically independent processes** if, for any finite set of time points

$$t_{11}, t_{12}, ..., t_{1n_1}, t_{21}, t_{22}, ..., t_{2n_2},$$

the vectors

$$\mathbf{X}_1 = (X_1(t_{11}), ..., X_1(t_{1,n_1})), \mathbf{X}_2 = (X_2(t_{21}), ..., X_2(t_{2,n_2})),$$

are independent.

2. A random variable $B = (B_1, ..., B_n) : [0, \infty) \to \mathbb{R}^n$ is an **n-dimensional Brownian motion** on the probability space $(\Omega, \mathcal{B}, P)$ if for each $i = 1, ..., n$, $B_i(t,\omega)$ is a 1-dimensional Brownian motion, and the stochastic processes $\{B_1, ..., B_n\}$ are stochastically independent.

As in Definition 1.4, it is easy to prove also in the multidimensional case (see [4]) that the process defined in definition 1.6 can be characterized as a process having stationary independent increments such that $X(t) - X(0)$ has an n-dimensional Gaussian distribution with zero mean and covariance matrix equal to $t \cdot \mathfrak{I}_n$, where $\mathfrak{I}_n$ is the n-dimensional identity matrix.

Moreover, letting $B'_i(t,\omega) = \alpha^{-1/2} B_i(\alpha t, \omega)$ we have $\forall x_i$, $u/\sqrt{\alpha} = v$ and $du = \sqrt{\alpha} v$:

$$\begin{aligned}
&P((B'_i(t+h,\omega) - B'_i(t,\omega)) \leq x_i) \\
&= P((B_i(\alpha(t+h),\omega) - B_i(\alpha t,\omega)) \leq \alpha^{1/2} x_i) \\
&= P((B_i(\alpha h,\omega) - B_i(0,\omega)) \leq \alpha^{1/2} x_i) \\
&= \alpha^{-1/2} \int_{-\infty}^{\alpha^{1/2} x_i} (2\pi h)^{-1/2} \exp(-\frac{{u_i}^2}{2\alpha h}) du_i \\
&= \int_{-\infty}^{x_i} (2\pi h)^{-1/2} \exp(-\frac{{v_i}^2}{2h}) dv_i \\
&= P((B_i(t+h,\omega) - B_i(t,\omega)) \leq x_i).
\end{aligned} \tag{1.7}$$

It is straightforward to show that $B'_i(t)$ is a continuous stochastic process starting from 0 and additive, and hence a Brownian motion. Since $\{B'_1, ..., B'_n\}$ are also independent, the process B' is an n-dimensional Brownian motion. Thus $\alpha^{1/2}B(t,\omega)$ and $B(\alpha t,\omega)$ have the same distribution, i.e. the Brownian paths are **self-similar**.

1.1 The statistical inversion method

We will use a mathematical formulation of the classical statistical inversion method as in [20, 21]. Usually, in solving ill-posed inverse problems, the quality of the result depends on how well one is able to make use of prior information. The main idea in the statistical inversion method is to consider the inverse problem as a Bayesian inference problem.

In statistical inversion theory, both the unknown quantity and the measurements are considered to be random variables. The randomness of the variables reflects the uncertainty on their actual value and the probability distribution of each variable describes its degree of uncertainty. The conditional distribution of the unknown variable given the measurement, the posterior density, will give us the solution in the Bayesian inference.

This last characteristic, in particular, makes the difference between the statistical approach and the traditional approach. Classical regularization methods produce single estimates for the unknown. The statistical method instead gives a distribution that can be used to obtain estimates of the unknown.

Consider measurable spaces $(M_i, \mathcal{B}_i)$, $i = 0, ..., n$. Let $m_i : \Omega \to M_i$ be a set of random variables such that $m_1, ...m_n$ are independent of m_0. Suppose that we want to retrieve the variable m_0 and denote the measurement vector by $(\hat{m}_1, ..., \hat{m}_n)$. Assuming that the conditional density $D(m_0, \hat{m}_1, ..., \hat{m}_n)$ exists, the **posterior distribution** of the variable m_0 can be defined by the conditional distribution

$$\begin{aligned} D_{post}(m_0) &= D(m_0|m_1 = \hat{m}_1, ..., m_n = \hat{m}_n) \\ &= \frac{D(m_0, \hat{m}_1, ..., \hat{m}_n)}{\int_{M_0} D(m_0, \hat{m}_1, ..., \hat{m}_n) dm_0} \\ &\sim D_{pr}(m_0)\, D(\hat{m}_1, ..., \hat{m}_n|m_0) \end{aligned} \tag{1.8}$$

where $D_{pr}(m_0)$ represents the prior distribution of m_0.

The approximate equal sign $\sim$ means that Equation (1.8) is true up to a normalization constant.

To estimate the point m_0 we use in this work one of the most popular statistical estimators, the **maximum a posteriori estimate** (MAP). Given the posterior probability density $D(m_0|m_1 = \hat{m}_1, ..., m_n = \hat{m}_n)$ of the unknown $m_0 \in M_0$, the MAP estimate will be

$$(m_0)_{MAP} = \arg \max_{m_0 \in M_0} D(m_0|m_1 = \hat{m}_1, ..., m_n = \hat{m}_n),$$

provided that such a maximizer exists.

In this work, we use the statistical inversion method in the study of the posterior distributions. In particular, the solution of the problem presented above is given by the point m_0 where the posterior distribution $D_{post}(m_0)$ maximizes, i.e. we calculate *the maximum likelihood estimate.*

One of the main problems is to specify the prior distribution of the unknown variable m_0. Usually, the only solution to this problem is to guess the most suitable prior distribution for m_0. In many cases, and in particular in this work, the prior distribution may be supposed to be constant.

1.2 The inverse problem

In the following we consider the stochastic process

$$X(t,\omega) = \sqrt{\hat{a}} B(t,\omega). \tag{1.9}$$

The positive constant $\hat{a}$ is called the amplitude parameter. Figure 1.1 shows a realiza-

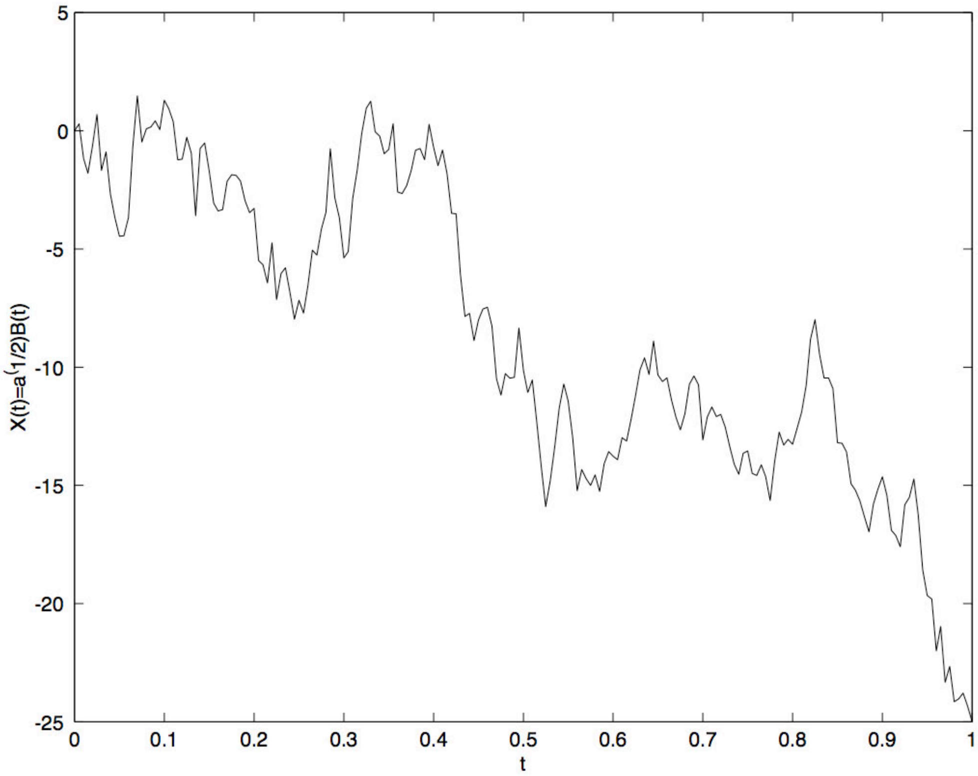

Figure 1.1: $X(t,\omega) = \sqrt{\hat{a}} B(t,\omega)$.

tion of the process defined above. In this section, our goal is to retrieve the amplitude parameter, if $X(t)$ is known for each $t \in I$. We use the statistical inversion method to find the posterior distribution of the amplitude parameter $\hat{a}$, and obtain the maximum likelihood estimate of our parameter.

Let $\mathbf{X}_{(n)} = (X(t_1, \omega), ..., X(t_n, \omega))^T$ be the vector of the observed values of (1.9) at n time instants $t_1, ..., t_n$. Since $X(t_i)$ is Gaussian, the joint distribution function of $\mathbf{X}_{(n)}$ is

$$D(X(t_1), ..., X(t_n)) = (2\pi)^{-\frac{n}{2}} \left|\Sigma_{(n)}\right|^{-\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{X}_{(n)}\Sigma_{(n)}^{-1}\mathbf{X}_{(n)})$$

where $\Sigma_{(n)}$ is the covariance matrix of the stochastic vector, i.e. (c.f. [17])

$$(\Sigma_{(n)})_{ij} = \hat{a} \cdot \min(t_i, t_j) \qquad \forall i, j = 1, ..., n.$$

In this paper we always assume that the stochastic process is non-degenerate, i.e. $\left|\Sigma_{(n)}\right| \neq 0$.

By Equation (1.8), the posterior distribution for the process (1.9) will be

$$D_{post}(\hat{a}, \mathbf{X}_{(n)}) = D_{pr}(\hat{a})\, D(\mathbf{X}_{(n)}|\hat{a}).$$

We also assume that we do not have any prior information on the behavior of $\hat{a}$. We always assume that

$$D_{pr}(a) = \frac{1}{t_n - t_1}.$$

In order to simplify notations, we define $\beta = (t_n - t_1)^{-1}$, but keep in mind that it depends on the length of the sampling interval. Thus for the process (1.9) we have

$$D_{post}(\hat{a}, \mathbf{X}_{(n)}) = \beta\, (2\pi)^{-n/2} \left|\Sigma_{(n)}\right|^{-1/2} \exp\big(-\frac{1}{2}\mathbf{X}_{(n)}^T\Sigma_{(n)}^{-1}\mathbf{X}_{(n)}\big). \tag{1.10}$$

Figure 1.2 shows a possible posterior distribution. In Bayesian inference, the study of this posterior distribution gives an estimate of $\hat{a}$.

In this way, the problem of recovering the amplitude parameter is solved. However, if we want to recover a parameter related to a physical phenomenon, we are dealing with a process defined for all $t \in I$, not just on a discrete set of observation times t_i. Once we make measurements, we make it a discrete problem. In many cases, it is still an open question whether the solution of the discretized problem will coincide with the solution of the continuous problem.

We prove that, in the limit $n \to \infty$, the solution of the discretized problem is the same as in the continuous case for the process defined in Equation (1.9).

1.3 Study of $D_{post}(\hat{a}, \mathbf{X})$ when $n \to \infty$ for $t \in [0, \infty)$

In this section we prove that, for the problem in Equation (1.9), the estimate $\bar{a}$ obtained from the posterior distribution coincides with the parameter $\hat{a}$ in the limit

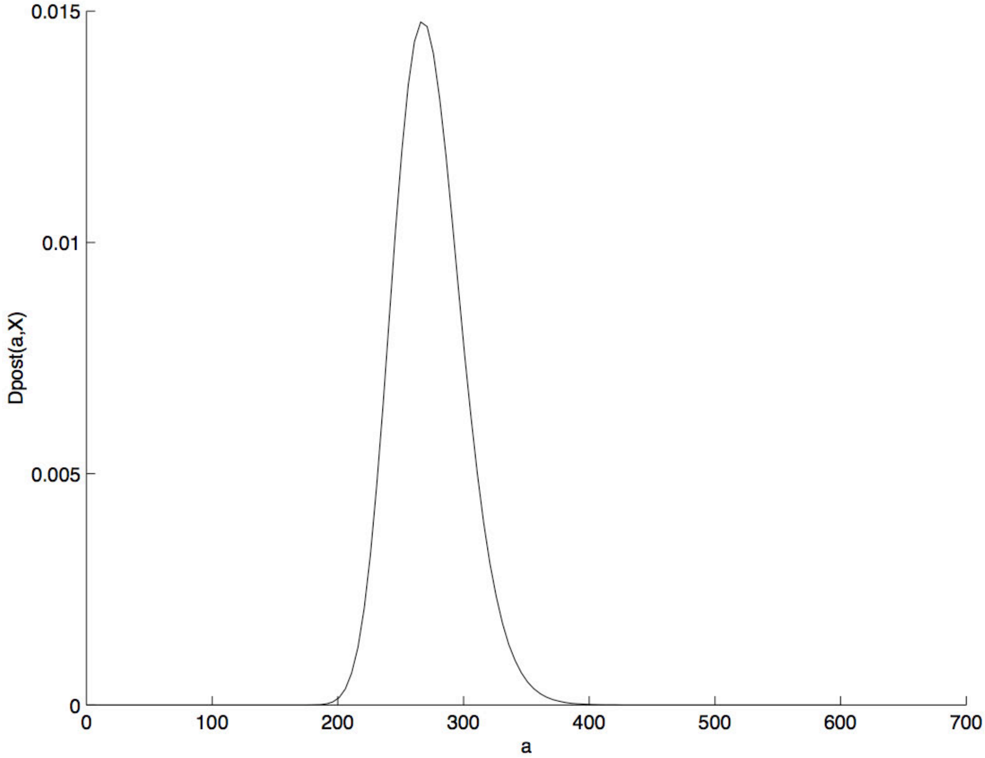

Figure 1.2: Posterior distribution $D_{post}(\hat{a}, \mathbf{X}_{(n)})$.

$n \to \infty$. This is possible when the posterior distribution is concentrated around the value $\hat{a}$.

Figure 1.3 shows how the width of the posterior distribution depends on the number of measurements. In particular, in the following calculation we assume equidistant measurement times, i.e. t_i are s.t. $t_1 < ... < t_n$ with $t_{i+1} - t_i = h$. By letting $n \to \infty$, we suppose that we are measuring for an infinitely long time. Although this is not possible in reality, it will be convenient to prove the theorem first in this case.

Before the proof, we need some preliminary results. We use Laplace's method to study the asymptotic behavior of integrals.

In the following, by $A \sim B$ as $x \to \infty$ we mean that

$$\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1.$$

In such case we say that A is asymptotic to B. Clearly this is a symmetric relation, i.e. $A \sim B \Longleftrightarrow B \sim A$.

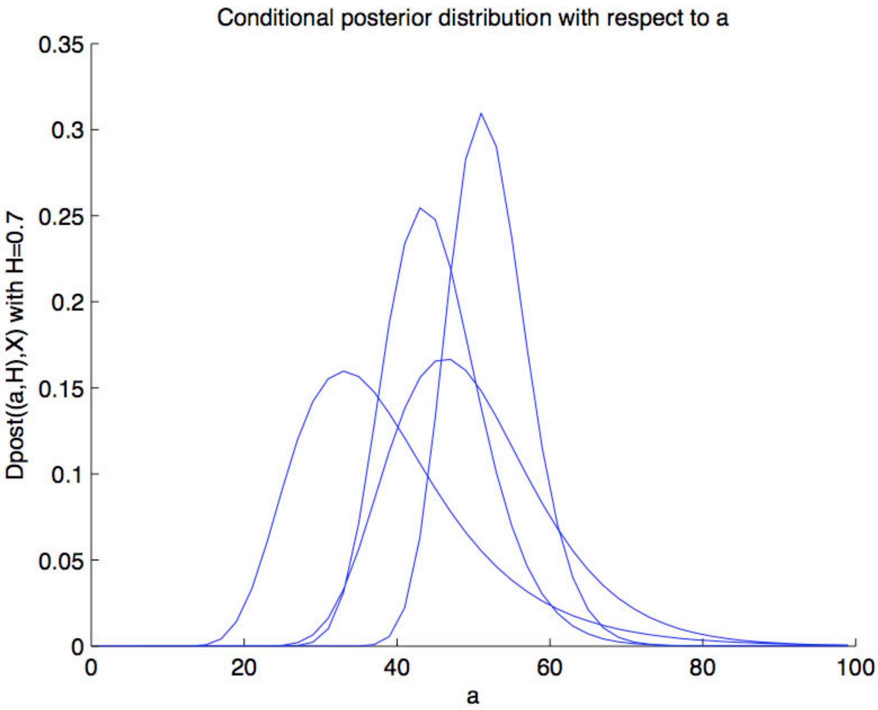

Figure 1.3: Posterior distributions of a realization of the same process with a different number of measurements.

From Theorem 19.3b in [9], we have the following corollary:

Lemma 1.11. *(Laplace's method) Suppose $f : [b, d] \to \mathbb{R}$ satisfies the following:*

1. *f has two continuous derivatives in $\{b < t < d\}$, where b and d may be finite or infinite,*
2. *f is decreasing in $\{b < t \leq c\}$ and*
   f is increasing in $\{c \leq t < d\}$
3. *$f'(c) = 0$ and $f''(c) > 0$,*
4. *there is an x_0 for which*
$$I(x) = \int_a^b e^{-xf(t)} dt$$
   exists for $x = x_0$.

Then $I(x)$ exists for all $x > x_0$ and

$$I(x) \sim \frac{e^{-xf(c)}\sqrt{2\pi}}{\sqrt{xf''(c)}} \qquad as \quad x \to \infty.$$

By Stirling's formula [9]

$$\Gamma(x+1) \sim \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} \qquad as \quad x \to \infty. \tag{1.12}$$

Let $\mathbf{X}_{(n)}$ be the process defined in (1.9). To simplify notations, we define $y_i = X(t_i, \omega) - X(t_{i-1}, \omega)$, with $t_0 = 0$.

From the definition of Brownian motion, it follows that the covariance matrix of the stochastic process $\mathbf{X}_{(n)}$ is equal to $\hat{a}\,\tilde{\Sigma}_{(n)}$ where

$$\tilde{\Sigma}_{(n)} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 \\ 1 & 2 & 3 & \dots & 3 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n-1 & n-1 \\ 1 & 2 & 3 & \dots & n-1 & n \end{pmatrix}, \tag{1.13}$$

and its inverse

$$\tilde{\Sigma}^{-1}_{(n)} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}. \tag{1.14}$$

Because of the simple structure of the inverse of the matrix $\tilde{\Sigma}_{(n)}$, we have

$$\mathbf{X}^T_{(n)} \tilde{\Sigma}^{-1}_{(n)} \mathbf{X}_{(n)} = \sum_{i=1}^{n} (X(t_i, \omega) - X(t_{i-1}, \omega))^2 = \sum_{i=1}^{n} y_i^2. \tag{1.15}$$

Denote $S_n = \sum_{i=1}^n y_i^2$. Note that from Definition 1.4 it follows that $y_i \sim \eta(0, \hat{a}\, h)$, where $h = t_{i+1} - t_i$. Without loss of generality, we may set $h = 1$. Also note that $\left|\Sigma_{(n)}\right| = \hat{a}^n \left|\tilde{\Sigma}_{(n)}\right|$. Then, since $\left|\tilde{\Sigma}_{(n)}\right| = 1$, the process $\mathbf{X}_{(n)}$ will be non-degenerate for all $\hat{a} \neq 0$.

We can now prove the following:

Theorem 1.16. *Let $\mathbf{X}_{(n)} = (X(t_1, \omega), ..., X(t_n, \omega))^T$ be the vector of measurements in* (1.9) *with $t_{i+1} - t_i = h$, and $\bar{a}$ a random variable with distribution $D_{post}(\hat{a}, X_{(n)})$ as given in (1.10). Then, for all $\varepsilon > 0$ such that $\varepsilon < \hat{a}$*

$$\lim_{n \to \infty} \mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = 1.$$

Proof. From Equations (1.10) and (1.15) we have

$$\mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = \mathbb{E}\big[\frac{\beta}{\mathbf{C}_{(n)} (S_n \pi)^{n/2}} \int_{\hat{a}-\varepsilon}^{\hat{a}+\varepsilon} (\frac{S_n}{2a})^{n/2} \exp(-\frac{S_n}{2a}) da\big]$$

where $\mathbf{C}_{(n)}$ is a normalization constant.

Substituting $t = S_n/2a$ this becomes

$$\mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = \frac{\beta}{2\pi}\,\mathbb{E}\big[\frac{1}{\mathbf{C}_{(n)}(S_n\pi)^{n/2-1}}\int_{b_n}^{d_n} t^{n/2-2}\,e^{-t}dt\big],$$

where

$$b_n = \frac{S_n}{2(\hat{a}+\varepsilon)} \qquad \text{and} \qquad d_n = \frac{S_n}{2(\hat{a}-\varepsilon)}.$$

The normalization constant

$$\begin{aligned}\mathbf{C}_{(n)} &= \int_0^\infty \frac{\beta}{2\pi a^{n/2}}\,\exp(-\frac{S_n}{2a})da \\ &= \frac{1}{2\pi(\pi S_n)^{n/2-1}}\,\Gamma(n/2-1).\end{aligned}$$

can be calculated analogously. Therefore, we have

$$\mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = \frac{1}{\Gamma(n/2-1)}\,\mathbb{E}\big[\int_{b_n}^{d_n} t^{n/2-2}e^{-t}dt\big], \tag{1.17}$$

where $\beta = (t_n - t_1)^{-1}$. Defining

$$I(N) = \int_{b_n}^{d_n} t^{n/2-2}\,e^{-t}dt$$

where $N = n/2 - 2$, we can rewrite Equation (1.17) as

$$\mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = \frac{1}{\Gamma(N+1)}\mathbb{E}[I(N)]. \tag{1.18}$$

Letting $t = Nu$ we get

$$\begin{aligned}I(N) &= \int_{b_n}^{d_n} t^{n/2-2}\,e^{-t}dt \\ &= \int_{b_n/N}^{d_n/N} e^{-uN}(uN)^N N du \\ &= \int_{b_n/N}^{d_n/N} N^{N+1}\,e^{-uN}\,u^N du \\ &= N^{N+1}\int_{b_n/N}^{d_n/N} e^{-N(u-\log u)}du.\end{aligned} \tag{1.19}$$

Since S_n is the sum of positive independent random variables with a mean value $\hat{a}$, it follows by the Law of Large Numbers [9] that

$$\lim_{n\to\infty}\frac{S_n}{n} = \hat{a}. \qquad a.s.$$

Next, we use Lemma 1.11 to calculate the integral (1.19). To do this, we estimate the interval $[b_n/N, d_n/N]$ in the limit for $n \to \infty$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{b_n}{N} &= \lim_{n\to\infty} \frac{S_n}{2\,N\,(\hat{a}+\varepsilon)} \\ &= \lim_{n\to\infty} \frac{n\hat{a}}{2(n/2-2)(\hat{a}+\varepsilon)} \\ &= \lim_{n\to\infty} \frac{n}{(n-4)(1+\frac{\varepsilon}{\hat{a}})} > \frac{1}{2} \qquad \forall \varepsilon < \hat{a}.\end{aligned}$$

Hence $[b_n/N, d_n/N] \subset [\frac{1}{2}, \infty)$ a.s. when n is large enough.

Next, we apply Lemma 1.11 with $f(u) = u - \log u$. The critical point is $c = 1$. Note also that $c \in [\frac{1}{2}, \infty)$. The function $f(u)$ behaves as

1. $f(u), \quad f'(u), \quad f''(u)$ are continuous functions $\forall u \in [\frac{1}{2}, \infty)$.
2. $f(u)$ is decreasing $\forall u \in [\frac{1}{2}, 1]$ and $f(u)$ is increasing $\forall u \in [1, \infty)$.
3. At the critical point $u = 1$, we have $f''(1) = 1 > 0$, i.e. the critical point is non-degenerate.

Since all hypotheses of Corollary 1.11 are satisfied, we obtain

$$I(N) \sim N^{N+1} e^{-Nf(1)} \frac{\sqrt{2\pi}}{\sqrt{Nf''(1)}} \qquad as \qquad N \to \infty.$$

Then, from Stirling's formula (1.12), we finally have the asymptotic behavior of the integral in Equation (1.18):

$$\begin{aligned}\lim_{n\to\infty} \frac{I(N)}{\Gamma(N+1)} &= \lim_{n\to\infty} \frac{N^{N+1} e^{-Nf(1)}}{N^{N+1/2} e^{-N} \sqrt{2\pi}} \frac{\sqrt{2\pi}}{\sqrt{Nf''(1)}} \\ &= \lim_{n\to\infty} \frac{N^{1/2} e^{-N(1-\log 1)}}{e^{-N}\sqrt{N \cdot 1}} = 1. \qquad a.s.\end{aligned}$$

The use of Lebesgue's dominated convergence theorem leads to

$$\lim_{n\to\infty} \mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = \mathbb{E}[\lim_{n\to\infty} \frac{I(N)}{\Gamma(N+1)}] = 1.$$

The theorem is proved. ∎

This theorem proves that the probability mass is all concentrated on the point $\hat{a}$ when we measure the phenomena at the equidistant instants $t_1, ..., t_n$ and $n \to \infty$. In practice, this means that should be measuring for an infinite period of time.

1.4 Study of $D_{post}(\hat{a}, \mathbf{X})$ when $n \to \infty$ for $t \in [0,1]$

Next, we rewrite the results of the previous section in the physical case $t \in [0,1]$ and the sample points $\bar{t}_1, ..., \bar{t}_n \in [0,1]$. This is based on the self-similarity of BM.

Consider the stochastic process

$$X(\bar{t}, \omega) \sim X(\frac{t}{n}, \omega).$$

Then, from self-similarity follows

$$X(\bar{t}, \omega) \sim \sqrt{\frac{1}{n}} X(t, \omega).$$

The previous theorem can be reformulated as

Theorem 1.20. *Let* $\bar{\mathbf{X}}_{(n)} = (X(\bar{t}_1, \omega), ..., X(\bar{t}_n, \omega))^T$ *be the measurement vector in (1.9) with* $\bar{t}_{i+1} - \bar{t}_i = \frac{h}{n}$*, and* $\bar{a}$ *a random variable with the distribution* $D_{post}(\hat{a}, X_{(n)})$ *as given by Equation* (1.10)*. Then, for all* $\varepsilon > 0$ *such that* $\varepsilon < \hat{a}$

$$\lim_{n\to\infty} \mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = 1.$$

Proof. The proof of this theorem is very similar to the previous one, the crucial point being to pay attention to the fact that the process is scaled. We note some consequences of the self-similarity for the stochastic process $\bar{\mathbf{X}}_{(n)}$ with respect to the process $\mathbf{X}_{(n)}$.

1. $\left|\bar{\Sigma}_{(n)}\right| = (\frac{1}{n})^n \left|\Sigma_{(n)}\right| = (\frac{\hat{a}}{n})^n$
2. $(\bar{\Sigma}_{(n)})^{-1} = n\,(\Sigma_{(n)})^{-1} = \frac{n}{\hat{a}}(\tilde{\Sigma}_{(n)})^{-1}$
3. $\bar{S}_n = \mathbf{X}_{(n)}\,\tilde{\Sigma}_{(n)}^{-1}\,\mathbf{X}_{(n)}^T$

where $\Sigma_{(n)}$ and $\bar{\Sigma}_{(n)}$ are the covariance matrices of the processes $\mathbf{X}_{(n)}$ and $\bar{\mathbf{X}}_{(n)}$, respectively, and $\bar{\Sigma}_{(n)} = \frac{\hat{a}}{n}\tilde{\Sigma}_{(n)}$.

With the above equalities and the same notation as in theorem 1.16, we can rewrite Equation (1.19) as:

$$I(N) = N^{N+1} \int_{b_n^*}^{d_n^*} e^{-N(u - \log u)} du,$$

where

$$b_n^* = \frac{\bar{S}_n\, n}{2Nt\,(\hat{a} + \varepsilon)} \qquad \text{and} \qquad d_n^* = \frac{\bar{S}_n\, n}{2Nt\,(\hat{a} - \varepsilon)}.$$

Remembering that $\bar{S}_n$ is the sum of positive independent random variables with a mean value $\hat{a}/n$, in the limit for $n \to \infty$ we have

$$\begin{aligned}\lim_{n\to\infty} b_n^* &= \lim_{n\to\infty} \frac{n\bar{S}_n}{2N\left(\hat{a}+\varepsilon\right)} \\ &= \lim_{n\to\infty} \frac{n\hat{a}}{2\left(n/2-2\right)\left(\hat{a}+\varepsilon\right)} > \frac{1}{2} \qquad \forall \varepsilon < \hat{a}.\end{aligned}$$

Hence $[b_n^*, d_n^*] \subset [\frac{1}{2}, \infty)$ a.s. when n is large enough.

Proceeding in the same way as in Theorem 1.16, the claim follows. ■

Chapter 2

Retrieving either underlying parameter of fractional Brownian motion

The fractional Brownian motion (FBM) with a Hurst parameter $H \in (0,1)$ was introduced in [16] as a centered Gaussian process $Z_H = \{Z_H(t), t \geq 0\}$ with a covariance

$$\mathbb{E}(Z_H(s), Z_H(t)) = (\Sigma_H)_{st} = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{2.1}$$

It is a process starting from zero almost surely. It has stationary increments, $\mathbb{E}(Z_H(t) - Z_H(s))^2 = |t-s|^{2H}$, is self-similar, i.e. $Z_H(\alpha t)$ has the same distribution as $\alpha^H Z_H(t)$. The value of H determines the nature of the FBM:

1. if $H = \frac{1}{2}$, the process is a regular Brownian motion.
2. if $H < \frac{1}{2}$, the increments of the process are positively correlated.
3. if $H > \frac{1}{2}$, the increments of the process are negatively correlated.

In this chapter, we consider a stochastic process X with an amplitude parameter $\hat{a}$ such that

$$X(t,\omega) = \sqrt{\hat{a}} Z_H(t,\omega). \tag{2.2}$$

The basic properties of this process are

1. $E(X(t)) = 0$ for all $t \geq 0$
2. $Var(X(t)) = \hat{a}\,|t|^{2H}$ for all $t \geq 0$
3. $E[(X(t+h) - X(h))(X(s+h) - X(h))] = \frac{\hat{a}}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$, for all $t, s \geq 0$.

The main task in this chapter is to generalize the results obtained in the previous chapter. In the case of FBM, we cannot use the law of large numbers, since the realizations are not independent. However, we will try to find a similar instrument to prove that when $H \in (0,1)$ is fixed, the mass of the posterior distribution is asymptotically concentrated on $\hat{a}$ also for the process (2.2).

2.1 The posterior distribution of the amplitude parameter $\hat{a}$

In this section, we assume that the value of the Hurst parameter H is fixed and study the posterior distribution of the amplitude parameter $\hat{a}$.

Let $\mathbf{X}_{(n)} = (X(t_1,\omega),...,X(t_n,\omega))^T$ be the vector of observed values of the process defined in (2.2) at n time instants $t_1,...,t_n$. The conditional distribution of the process $\mathbf{X}_{(n)}$, given that the random variable a is equal to $\hat{a}$, is

$$D_{post}(\hat{a},\mathbf{X}_{(n)}) = D_{pr}(\hat{a})D(\mathbf{X}_{(n)}|\hat{a}). \tag{2.3}$$

Since $X(t_i)$ is Gaussian and we suppose the prior distribution of the amplitude parameter $\hat{a}$ is a constant c, the posterior distribution equals

$$D_{post}(\hat{a},\mathbf{X}_{(n)}) = \beta\,(2\pi)^{-n/2}\left|(\Sigma_H)_{(n)}\right|^{-1/2}\exp(-\frac{1}{2}\mathbf{X}_{(n)}^T(\Sigma_H)_{(n)}^{-1}\mathbf{X}_{(n)}), \tag{2.4}$$

where $\beta = (t_n - t_1)^{-1}$.

To prove a generalization of Theorem 1.20, we proceed in the same way as we did in the previous chapter. However, since the increments of FBM are not independent as was the case of BM, the covariance matrix $(\Sigma_H)_{(n)}$ is more complicated. In particular, the quantity $\mathbf{X}^T(\Sigma_H)_{(n)}^{-1}\mathbf{X}$, necessitates some additional work.

To solve this problem, we introduce the process called *fractional Gaussian noise*, fGn, in the following way.

Let $\mathbf{Y}_{(n)} = (Y_1,...,Y_n)^T$ be the realization of the stochastic process defined by

$$Y_i = X(t_i) - X(t_{i-1}) \tag{2.5}$$

where $t_i = \dfrac{i}{n}$ and $t_0 = 0$ with $i = 0,...,n$.

The process $\mathbf{Y}_{(n)}$ is a strongly correlated stationary sequence. In particular, for $H > \frac{1}{2}$, the correlation between the past and future is positive, increasing from 0 to 1 as H increases from $\frac{1}{2}$ to 1. This means that a fGn with $H > \frac{1}{2}$ is persistent and the persistence increases with H. On the contrary, in the case $H < \frac{1}{2}$, this correlation is negative and decreases from 0 to $-\frac{1}{2}$ as H decreases from $\frac{1}{2}$ to 0. This means that a fGn with $H < \frac{1}{2}$, large positive values tend to be followed by large negative values and vice versa.

It is not difficult to calculate from Equation (2.1) the covariance matrix associated with the process $\mathbf{Y}_{(n)}$. For all $i, j = 1, ..., n$ we have:

$$\big((\Gamma_H)_{(n)}\big)_{ij} = \frac{1}{2}\big(\,|i-j+1|^{2H} + |i-j-1|^{2H} - 2\,|i-j|^{2H}\,\big). \tag{2.6}$$

Note that $Y_i \sim \eta(0, \dfrac{\hat{a}}{n^{2H}})$, giving

$$D_{post}(\hat{a}, \mathbf{Y}_{(n)}) = \beta\,(\frac{2\pi\hat{a}}{n^{2H}})^{-n/2}\,\big|(\Gamma_H)_{(n)}\big|^{-1/2} \exp\big(-\frac{n^{2H}}{2\hat{a}}\mathbf{Y}^T_{(n)}(\Gamma_H)^{-1}_{(n)}\mathbf{Y}_{(n)}\big). \tag{2.7}$$

Clearly this posterior distribution contains exactly the same information about the parameter $\hat{a}$ as the posterior distribution in (2.4). Since the stochastic process $\mathbf{Y}_{(n)}$ is stationary, it will be more convenient in the calculation of the posterior distribution.

2.2 The inverse of $(\Gamma_H)_{(n)}$

First of all, note that since the stochastic process $\mathbf{Y}_{(n)}$ is stationary, the covariance matrix $(\Gamma_H)_{(n)}$ is a Toeplitz matrix, i.e. a matrix of the form

$$[\gamma_{j-k}]^n_{j,k=0}.$$

In this section, we calculate the inverse of $(\Gamma_H)_{(n)}$ explicitly. We prove that the inverse of the covariance operator does exist and then use a result presented in [11] to find a suitable representation for the elements of the covariance matrix. On the basis of a numerical simulation, we construct a new matrix approximating the inverse of the covariance matrix and calculate its elements.

We prove that $(\Gamma_H)_{(n)}$ is invertible. To this aim, we first consider the infinite matrix Γ_H associated with the covariance operator and prove it is positive. Using the fact that $(\Gamma_H)_{(n)}$ is nothing else than the $n \times n$ upper-left corner of Γ_H, we prove that $(\Gamma_H)_{(n)}$ is invertible.

First, define $Z = \{z \in \mathbb{C}^{\mathbb{N}} | (z_j 2^j) \in l^\infty\}$ and $\mathbb{C}^{\mathbb{N}} = \{(z_j)^\infty_{j=1}; z_j \in \mathbb{C}\}$ and consider Γ_H to be a linear map $Z \to \mathbb{C}^{\mathbb{N}}$. Then

Proposition 2.8. *Γ_H is positive, i.e. $\langle \Gamma_H z, z\rangle > 0$ if $z \neq 0$.*

Proof. To prove that Γ_H is positive, we can study the spectral density. We need to prove that $\langle \Gamma_H z, z\rangle > 0$ if the real vector $z \in Z$ is not a zero vector.

To this aim, we rewrite the element in (2.6) as:

$$(\Gamma_H)_{jk} = \gamma_{j-k} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i(j-k)\lambda} f(\lambda) d\lambda \tag{2.9}$$

where

$$f(\lambda) = C \left|e^{2\pi i\lambda} - 1\right|^2 \sum_{m=-\infty}^{\infty} \frac{1}{|\lambda + m|^{2H+1}} \qquad -\frac{1}{2} \leq \lambda \leq \frac{1}{2}, \quad \lambda \neq 0,$$

with $C > 0$, is the spectral density of fGn (see [19]). Noting that Γ_H is symmetric, since f is even, and

$$\left|e^{2\pi i\lambda} - 1\right|^2 = \left|2\pi i\lambda + O(|\lambda|^2)\right|^2 = 4\pi |\lambda|^2 + O(|\lambda|^3) \qquad as \quad |\lambda| \to \infty,$$

we see that this spectral density is proportional to $|\lambda|^{1-2H}$ near $\lambda = 0$.

Therefore, the spectral density is continuous, if $H < \frac{1}{2}$. Moreover, the spectral density is singular but integrable at $\lambda = 0$, if $H > \frac{1}{2}$.

Hence, we can write:

$$\begin{aligned}\langle \Gamma_H z, z\rangle &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (\sum_{j=0}^{\infty} e^{2\pi ij\lambda} z_j)(\sum_{k=0}^{\infty} e^{-2\pi ik\lambda} z_k) f(\lambda) d\lambda \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sum_{j=0}^{\infty} e^{2\pi ij\lambda} z_j\right|^2 f(\lambda) d\lambda = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(\lambda)|^2 f(\lambda) d\lambda.\end{aligned} \tag{2.10}$$

The function

$$g(\lambda) = \sum_{j=0}^{\infty} e^{2\pi ij\lambda} z_j = \sum_{j=0}^{\infty} z_j w^j$$

is an analytic function on the disc and is not identically zero, and hence is equal to zero at most on a countable number of points.

Since $f(\lambda) > 0$ for $\lambda \neq 0$, and since $g(\lambda) = 0$ at most on a set of Lebesgue measure equal to zero, the integral in (2.10) is positive for each $z \neq \bar{0}$, i.e. (Γ_H) is positive. ■

Corollary 2.11. $(\Gamma_H)_{(n)}$ *is invertible.*

Proof. It is enough to show that the matrix $(\Gamma_H)_{(n)}$ defines an injective linear map $\mathbb{C}^n \to \mathbb{C}^n$. Assume $z = (z_1, ..., z_n)$ is such that $(\Gamma_H)_{(n)} z = 0$. Since $(\Gamma_H)_{(n)}$ is the $n \times n$ upper-left corner of (Γ_H), interpreting $z = (z_1, ..., z_n, 0, ..., 0) \in Z$, we obtain $\langle (\Gamma_H)_{(n)} z, z\rangle = \langle \Gamma_H z, z\rangle = 0$, i.e. $z = 0$. ■

To calculate the inverse of the covariance matrix, as already mentioned, we are going to use a result presented in [11]. Since we do not use the theorem in the form presented there, but only a part of it, we need some preliminary considerations.

Consider the system of equations

$$\sum_{k=0}^{n} \gamma_{j-k} x_k = \delta_{j0} \qquad (j = 0, ..., n), \tag{2.12}$$

which can be rewritten as

$$(x_0, x_1, ..., x_n)^T = (\Gamma_H)^{-1}_{(n)}(1, 0, ..., 0)^T \tag{2.13}$$

and since $(\Gamma_H)_{(n)}$ is invertible, the elements $(x_0, x_1, ..., x_n)$ can be determined uniquely. To get a factorization of the covariance matrix, we will use the following result

Theorem 2.14. *If a Toeplitz matrix $A_n = |a_{j-k}|^n_{j,k=0}$ satisfies*

$$\begin{cases} \sum_{k=0}^n a_{j-k}x_k = \delta_{j0} \qquad (j = 0, ..., n) \\ \sum_{k=0}^n a_{j-k}y_{k-n} = \delta_{jn} \qquad (j = 0, ..., n) \end{cases}$$

with $x_0 \neq 0$, then A_n is invertible and its inverse is

$$A_n^{-1} = x_0^{-1}\left\{\begin{pmatrix} x_0 & 0 & \dots & 0 \\ x_1 & x_0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ x_n & x_{n-1} & \dots & x_0 \end{pmatrix}\begin{pmatrix} y_0 & y_{-1} & \dots & y_{-n} \\ 0 & y_0 & \dots & y_{-n+1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & y_0 \end{pmatrix}\right.$$

$$\left. - \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ y_n & 0 & 0 & \dots & 0 & 0 \\ y_{-n+1} & y_{-n} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_{-1} & y_{-2} & y_{-3} & \dots & y_{-n} & 0 \end{pmatrix}\begin{pmatrix} 0 & x_n & x_{n-1} & \dots & x_1 \\ 0 & 0 & x_n & \dots & x_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x_n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}\right\}$$

Proof. See [11]. ∎

To prove that the matrix $(\Gamma_H)_{(n)}$ satisfies the hypothesis of Theorem 2.14 in addition to Equation (2.13), it is sufficient to prove that for the process in Equation (2.5), the element x_0 is different from zero. Suppose, by contradiction, that $x_0 = 0$, and rewrite Equation (2.12) as

$$\begin{cases} x_1\gamma_1 + x_2\gamma_2 + ... + x_n\gamma_n = 1 \\ x_1\gamma_0 + x_2\gamma_0 + ... + x_n\gamma_{n-2} = 0 \\ \dots \\ x_1\gamma_n + x_2\gamma_{n-1} + ... + x_n\gamma_0 = 0 \end{cases}$$

This is a system with $n + 1$ equations and n unknowns. It is enough to consider the last n equations to solve the system. We have

$$(x_1, ..., x_n)^T = (\Gamma_H)^{-1}_{(n-1)}(0, ..., 0)^T.$$

Thus, the fact that $x_0 = 0$ implies that the vector $(x_0, x_1, ..., x_n)$ is a null vector. This is a contradiction.

Since $(\Gamma_H)_{(n)}$ is symmetric, $\gamma_j = \gamma_{-j}$. This implies that $x_i = y_{-i} \quad (i = 0, ..., n)$ in the hypothesis of the above theorem. Then, since the system of Equations (2.13) is satisfied and $x_0 \neq 0$, the assumptions of Theorem 2.14 are satisfied.

From the factorization in Theorem 2.14 it follows that the elements of $(\Gamma_H)^{-1}_{(n)}$ can be calculated for all $j, k = 1, ...n$ from the formula (see [11]):

$$\big((\Gamma_H)^{-1}_{(n)}\big)_{jk} = g^{(n)}_{jk} = g^{(n)}_{j-1,k-1} + x_0^{-1}\big[x_j\, x_k - x_{n+1-j}\, x_{n+1-k}\big] \tag{2.15}$$

where

$$g^{(n)}_{0k} = x_k \qquad g^{(n)}_{k0} = x_k \qquad (k = 0, 1, ..., n).$$

Now that we have a recursive formula for $g^{(n)}_{jk}$, he following result on the asymptotic behavior of $g^{(\infty)}_{jk}$ as $k \to \infty$ can be stated:

Proposition 2.16. *With j fixed and $\frac{1}{2} < H < 1$*

$$\big((\Gamma_H)^{-1}\big)_{jk} = g^{(\infty)}_k = H(2H-1)\,|k|^{2H-2} + O(\frac{1}{k^2}) \qquad as \quad k \to \infty.$$

Proof. As shown in the proof of Proposition 2.8, the function $f(\lambda)$ has an integrable singularity of the type $|\lambda|^{1-2H}$ at origin.

Since j is fixed, we can suppose without loss of generality that $j = 0$. Then

$$\big((\Gamma_H)^{-1}\big)_{0k} = g^{(\infty)}_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-2\pi i k\lambda}}{f(\lambda)} d\lambda.$$

Using the result shown in the proof of Proposition 2.8, we can write the asymptotic behavior of the spectral density when $|\lambda| \to 0$:

$$\begin{aligned}
f(\lambda) &= C\left|e^{2\pi i\lambda} - 1\right|^2 \sum_{m=-\infty}^{\infty} \frac{1}{|\lambda + m|^{2H+1}} \\
&= C\left|e^{2\pi i\lambda} - 1\right|^2 \big(\,|\lambda|^{-2H-1} + \sum_{m\neq 0} \frac{1}{|\lambda + m|^{2H+1}}\big) \\
&= C(4\pi\,|\lambda|^2 + O(|\lambda|^3)\big(\,|\lambda|^{-2H-1} + \sum_{m\neq 0} \frac{1}{|\lambda + m|^{2H+1}}\big) \\
&= C\big(\,|\lambda|^{-2H+1} + \sum_{m\neq 0} \frac{|\lambda|^2}{|\lambda + m|^{2H+1}}\big) r(\lambda),
\end{aligned}$$

where $r(\lambda) = 4\pi + O(|\lambda|^3)$ as $|\lambda| \to \infty$.

Letting $R(\lambda) = 1/Cr(\lambda)$, we have

$$\begin{aligned} \frac{1}{f(\lambda)} &= \frac{1}{Cr(\lambda)(|\lambda|^{-2H+1} + \sum_{m\neq 0} \frac{|\lambda|^2}{|\lambda+m|^{2H+1}})} \\ &= \frac{R(\lambda)}{|\lambda|^{-2H+1}}\Big(\frac{1}{1+\phi(|\lambda|)}\Big) \end{aligned}$$

where

$$\phi(|\lambda|) = \sum_{m\neq 0} \Big(\frac{|\lambda|}{|\lambda+m|}\Big)^{2H+1}.$$

Remark 2.17. Noting that $\phi(|\lambda|) = O(|\lambda|^{2H+1})$ as $|\lambda| \to 0$ we have

- If $H > \frac{1}{2}$ the function $\phi(|\lambda|)$ is twice differentiable at the origin.
- If $H < \frac{1}{2}$ the function $\phi(|\lambda|)$ is once differentiable at the origin.

Thus

$$\begin{aligned} g_k^{(\infty)} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-2\pi i(k)\lambda}}{f(\lambda)} d\lambda \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi ik\lambda} |\lambda|^{2H-1} \Phi(|\lambda|) d\lambda, \end{aligned} \tag{2.18}$$

where

$$\Phi(|\lambda|) = \frac{R(\lambda)}{1+\phi(|\lambda|)}.$$

Since $2H+1 \leq 3$ for each $H \in (0,1)$, we have

$$\Phi(|\lambda|) = \frac{1}{4\pi C + O(|\lambda|^{2H+1})} \qquad as\, |\lambda| \to 0.$$

To prove the theorem, one has to study the asymptotic behavior of the integral in (2.18) as $k \to \infty$. Using the result in Proposition 2.19 presented below and Equation (2.18), it follows that for $\frac{1}{2} < H < 1$

$$\begin{aligned} g_k^{(\infty)} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(k)}\lambda\, |\lambda|^{2H-1} \Phi(|\lambda|^{-2H-1}) d\lambda \\ &= \frac{1}{2\pi C} \Gamma(2H) e^{i\pi H} |k|^{-2H} + O(|k|^{-2H-1}) \qquad as \qquad k \to \infty. \end{aligned}$$

This proves the theorem. ∎

Proposition 2.19. *(Ederlyi's theorem) If $g(\lambda) \in C^2$ and $0 < \beta < 1$ then:*

$$\int_0^a |\lambda|^{\beta-1} g(\lambda) e^{i\gamma x^{\alpha}} d\lambda \sim a_0 \gamma^{-\frac{\beta}{\alpha}} + O(\gamma^{-\frac{1+\beta}{\alpha}}) \qquad \gamma \to \infty$$

where

$$a_j = \frac{g^{(j)}(0)}{j!\alpha} \Gamma(\frac{j+\beta}{\alpha}) \exp(\frac{i\pi(j+\beta)}{2\alpha}).$$

Proof. See for example [6], page 157. The proof presented in [6] is for the case $g(\lambda) \in C^{\infty}$. In going through the proof of Theorem 2.19, we note that if $g(\lambda)$ is at least twice differentiable at the origin, we can write the term a_0 and give an estimate of the behavior of the remainder as $\gamma \to \infty$. ■

Remark 2.20. The proof of proposition 2.16 is usually given using the Tauberian theorem. We prove that neither in that case we can relax the restriction on the Hurst parameter H.

To do this, consider the function $h(x) = (1-x)^{2H} - 2 + (1+x)^{2H}$. Calculating the Taylor series for $h(x)$ around $x = 0$ we have

$$h(x) = 2H(2H-1)x^2 + O(x^2).$$

Letting $x = 1/k$ with

$$\gamma_k = \frac{1}{2} k^{2H} h(\frac{1}{k}), \qquad k \geq 1$$

and we have

$$\gamma_k = H(2H-1)\,|k|^{2H-2} + O(\frac{1}{k^2}), \qquad as \quad k \to \infty.$$

Applying the Tauberian theorem shown in [22] we can rewrite the integral in (2.18). However, to be able to use the Tauberian theorem, the process should have long-range dependence, i.e. $\frac{1}{2} < H < 1$.

Figure 2.1 shows a plot of the solution of Equation (2.13) calculated from the 90×90 covariance matrix of a realization of fGn measured over the time interval $[0, 1]$. Hence, from the computer simulations, it would seem that, for n large enough, only terms g_{j-k} with $|j-k| < 2$ contribute significantly to $\left((\Gamma_H)^{-1}_{(n)}\right)$. For this reason, we shall henceforth consider the following approximate problem:

Given the posterior distribution

$$D^{app}_{post}(\hat{a}, \mathbf{Y}_{(n)}) = \beta(\frac{2\pi\hat{a}}{n^{2H}})^{-n/2} \left|(\Gamma_H)_{(n)}\right|^{-1/2} \exp(-\frac{n^{2H}}{2\hat{a}} \mathbf{Y}^T_{(n)} (\Gamma_H)^{-1}_{(n),app} \mathbf{Y}_{(n)}), \quad (2.21)$$

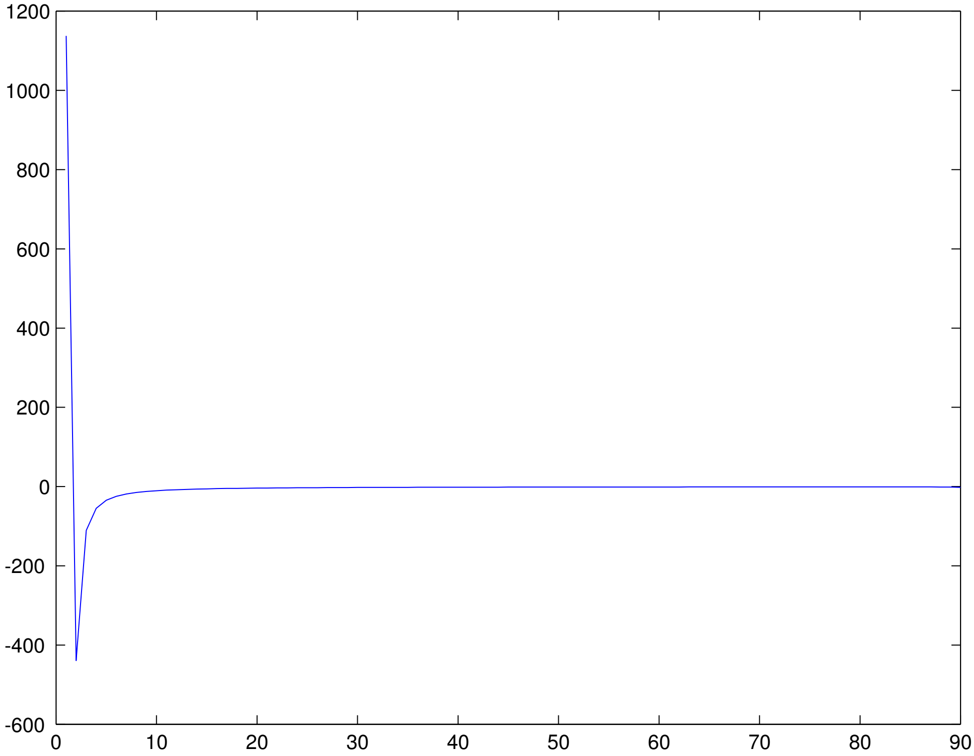

Figure 2.1: Solution vector of Equation (2.13).

where

$$(\Gamma_H)^{-1}_{(n),app} = \begin{pmatrix} c_1 & c_2 & 0 & \cdots & \cdots & \cdots \\ c_2 & c_1 & c_2 & 0 & \cdots & \cdots \\ 0 & c_2 & c_1 & c_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & c_2 & c_1 & c_2 \\ \cdots & \cdots & \cdots & 0 & c_2 & c_1 \end{pmatrix}, \tag{2.22}$$

we prove that $\lim_{n\to\infty} \mathbf{P}(\hat{a} - \varepsilon < \bar{a} < \hat{a} + \varepsilon) = 1$.

In the following, the elements of the matrix $(\Gamma_H)^{-1}_{(n),app}$ are calculated using the method illustrated in [23]. This paper gives a result which is theoretically quite similar to the one we found using Theorem 2.14, but simpler to apply numerically.

Suppose that $(\Gamma_H)^{-1}_{(n)} \approx (\Gamma_H)^{-1}_{(n),app}$. Thus $(\Gamma_H)^{-1}_{(n),app}(\Gamma_H)_{(n)} \approx \mathfrak{I}_{(n)}$, where $\mathfrak{I}_{(n)}$ is the n-dimensional identity matrix. This means

$$\begin{pmatrix} c_1 & c_2 & 0 & \cdots \\ c_2 & c_1 & c_2 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & c_2 & c_1 \end{pmatrix} \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

One way of calculating numerically the value of the elements c_i with $i = 1, 2$ is to consider the 3×3 upper-left corner of the full matrices $(\Gamma_H)_{(n)}$ and $(\Gamma_H)^{-1}_{(n),app}$ and, for example, to solve the equation

$$(c_2, c_1, c_2)[(\Gamma_H)_{(n)}]_{3\times 3} = (0, 1, 0).$$

In this way we get the explicit solution

$$c_i = ((\Gamma_H)^{-1}_{(n)})_{2,(2+i-1)}.$$

A 3×3 matrix is the smallest we can consider in the case of an approximate inverse covariance matrix with three diagonal elements different from zero. On the other hand, it is easily seen that it is not useful to consider a larger matrix in this case. If we, for example, consider a 4×4 matrix, then in each row we have at least one element equal to zero. If we consider the equation

$$(c_2, c_1, c_2, 0)[(\Gamma_H)_{(n)}]_{3\times 3} = (0, 1, 0, 0),$$

the only additional information we get from this solution respect to the previous one is that $c_3 = 0$. Since we suppose that we have only three diagonals different from zero, it is enough to consider the 3×3 matrix to calculate the value of the elements.

In the following, the calculation of c_i with $i = 1, 2$ in our problem is shown. We have

$$[(\Gamma_H)_{(n)}]_{3\times 3} = \begin{pmatrix} 1 & l(H) & b(H) \\ l(H) & 1 & l(H) \\ b(H) & l(H) & 1 \end{pmatrix}, \tag{2.23}$$

where

$$l(H) = 2^{2H-1} - 1 \tag{2.24}$$

and

$$b(H) = \frac{3^{2H} - 2^{2H+1} + 1}{2}. \tag{2.25}$$

Because of the simple structure of $[(\Gamma_H)_{(n)}]_{3\times 3}$, we can calculate inverse:

$$[(\Gamma_H)_{(n)}]^{-1}_{3\times 3} = \frac{1}{|[(\Gamma_H)_{(n)}]_{3\times 3}|} \begin{pmatrix} 1 - l(H)^2 & l(H)(b(H) - 1) & l(H)^2 - b(H) \\ l(H)(b(H) - 1) & 1 - b(H)^2 & l(H)(b(H) - 1) \\ l(H)^2 - b(H) & l(H)(b(H) - 1) & 1 - l(H)^2 \end{pmatrix} \tag{2.26}$$

where

$$\left|[(\Gamma_H)_{(n)}]_{3\times 3}\right| = (1-b(H))(1+b(H)-2l(H)^2). \tag{2.27}$$

Then we can write:

$$c_1(H) = \left([(\Gamma_H)_{(n)}]_{3\times 3}\right)_{2,2} = \frac{1+b(H)}{1+b(H)-2l(H)^2} \tag{2.28}$$

and

$$c_2(H) = \left([(\Gamma_H)_{(n)}]_{3\times 3}\right)_{2,3} = -\frac{l(H)}{1+b(H)-2l(H)^2}. \tag{2.29}$$

We now have explicit values for $c_1(H)$ and $c_2(H)$. This will allow us to calculate the exponential term $\mathbf{Y}_{(n)}^T(\Gamma_H)_{(n),app}^{-1}\mathbf{Y}_{(n)}$.

2.3 Convergence of the quadratic term S_n

Let $S_n = \mathbf{Y}_{(n)}^T(\Gamma_H)_{(n),app}^{-1}\mathbf{Y}_{(n)}$. Because of the simple structure of the approximation matrix $(\Gamma_H)_{(n),app}^{-1}$, we have

$$S_n = c_1 T_n + 2c_2 U_n$$

where

$$T_n = \sum_{i=1}^{n} Y_i^2 \tag{2.30}$$

and

$$U_n = \sum_{i=1}^{n-1} Y_i Y_{i+1}. \tag{2.31}$$

As already noted in the beginning of this chapter, the stochastic process $\mathbf{Y}_{(n)}$ is stationary, but the variables Y_i are not independent, even if they are identically distributed. More work is needed to study the convergence of the sum S_n.

We start by studying the asymptotic behavior of the sums T_n and U_n. To this aim, we apply the Birkhoff-Khinchin theorem (see [10]) to get a result which is similar to the strong law of large numbers for the process $\mathbf{Y}_{(n)}$.

Since $t_i = i/n$, every time a change in the value of n gives a new set on points. To avoid this problem, we need a new process.

Consider the process defined by

$$Y_i^{\inf} = X(\bar{t}_i) - X(\bar{t}_{i+1})$$

where $\bar{t}_i - \bar{t}_{i-1} = 1$ for $i = 1, ..., n$ and $\bar{t}_0 = 0$.

Next, define

$$T_n^{\inf} = \sum_{i=1}^{n} (Y_i^{\inf})^2 \tag{2.32}$$

and

$$U_n^{\inf} = \sum_{i=1}^{n} Y_i^{\inf} Y_{+1}^{\inf}. \tag{2.33}$$

Note that the sequences $T_n^{\inf}$ and $U_n^{\inf}$ are stationary, since $\mathbf{Y}_{(n)}^{\inf}$ is a stationary process.

The Birkhoff-Khinchin theorem can be applied to this process. Note that the time runs from 0 to infinity as $n \to \infty$. In particular, we will use the following corollary of the Birkhoff-Khinchin theorem:

Corollary 2.34. *Let $\{X_t\}$ denote a stationary sequence and suppose that $\mathbb{E}\{X_0\} < \infty$. Then, for any positive integer m, with probability 1 the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=n}^{n+m} X_k = X^*$$

exists, and $\mathbb{E}\{X^*\} = \mathbb{E}\{X_0\}$.

Proof. See [10]. ■

From Corollary 2.34 it follows that

$$\lim_{n\to\infty} \frac{T_n^{\inf}}{n} = T^{\inf}$$

and

$$\lim_{n\to\infty} \frac{U_n^{\inf}}{n-1} = U^{\inf}$$

with

$$\mathbb{E}(T^{\inf}) = \mathbb{E}((Y_0^{\inf})^2) = \mathbb{V}ar((Y_0^{\inf})) = \hat{a}$$

and

$$\mathbb{E}(U^{\inf}) = \mathbb{E}(Y_i^{\inf} Y_{i+1}^{\inf}) = \mathbb{C}ov(Y_i^{\inf} Y_{i+1}^{\inf}) = \hat{a}(2^{2H-1} - 1).$$

Since the process $\mathbf{Y}_{(n)}$ is H-self-similar, i.e. $Y(\alpha t)$ has the same distribution as $\alpha^{-H} Y(t)$, and $n^{-H} Y_i$ has the same distribution as $Y_i^{\inf}$.

Therefore, it follows that in the limit $n \to \infty$

$$\frac{T_n}{n} \sim T(n)$$

and

$$\frac{U_n}{n-1} \sim U(n)$$

with

$$\mathbb{E}(T(n)) = \mathbb{E}((Y_0)^2) = \mathbb{V}ar((Y_0)) = \frac{\hat{a}}{n^{2H}}$$

and

$$\mathbb{E}(U(n)) = \mathbb{E}(Y_iY_{i+1}) = \mathbb{C}ov(Y_i, Y_{i+1}) = \frac{\hat{a}}{n^{2H}}(2^{2H-1}-1).$$

Thus for the quadratic term S_n in the limit for $n\to\infty$, we have

$$\begin{aligned}\mathbb{E}(S_n) &= \mathbb{E}\big(c_1(H)T_n + 2c_2(H)U_n\big)\\ &\sim \big(c_1(H)\, n\, \mathbb{E}(T(n)) + 2c_2(H)(n-1)\mathbb{E}(U(n))\big)\\ &= \big(n\,\frac{\hat{a}}{n^{2H}}\, c_1(H) + 2\frac{\hat{a}(n-1)(2^{2H-1}-1)}{n^{2H}}c_2(H)\big)\\ &= \frac{\hat{a}}{n^{2H}}\frac{n(1+b(H))-(n-1)\, l(H)^2}{1+b(H)-2l(H)^2},\end{aligned} \tag{2.35}$$

where $l(H)$ and $b(H)$ are defined in (2.24) and (2.25).

2.4 Study of $D^{app}_{post}(\hat{a},\mathbf{Y})$ when $n\to\infty$ for $t\in[0,1]$

In the following we will prove a result analogous to Theorem 1.20 in the case of FBM. In the computations, instead of the inverse covariance matrix $(\Gamma_H)_{(n)}$, we use its approximation $(\Gamma_H)^{-1}_{(n),app}$.

Theorem 2.36. *Let $\mathbf{Y}_{(n)} = (Y_1,...,Y_n)^T$ be the measurement vector in Equation (2.5) with $\frac{1}{2} < H < 1$ and $t_{i+1}-t_i = \dfrac{h}{n}$, and $\bar{a}$ a random variable with distribution $D^{app}_{post}(\hat{a},\mathbf{Y}_{(n)})$ as given in Equation (2.21). Then for all $\varepsilon > 0$ such that $\varepsilon < \hat{a}$,*

$$\lim_{n\to\infty}\mathbf{P}(\hat{a}-\varepsilon < \bar{a} < \hat{a}+\varepsilon) = 1.$$

Proof. Starting from Equation (2.21), we have

$$\begin{aligned}&\mathbf{P}(\hat{a}-\varepsilon < \bar{a} < \hat{a}+\varepsilon)\\ &= \mathbb{E}\big[\frac{1}{\mathbf{C}_{(n)}}\int_{\hat{a}-\varepsilon}^{\hat{a}+\varepsilon}\frac{\beta n^{nH}}{(2\pi a)^{n/2}}\left|(\Gamma_H)_{(n)}\right|^{-\frac{1}{2}}\exp\!\big(-\frac{n^{2H}}{2a}\mathbf{Y}^T_{(n)}(\Gamma_H)^{-1}_{(n),app}\mathbf{Y}_{(n)}\big)da\big]\\ &= \frac{\left|(\Gamma_H)_{(n)}\right|^{-\frac{1}{2}}}{\mathbf{C}_{(n)}}\frac{\beta}{\pi^{n/2}}\mathbb{E}\big[\int_{\hat{a}-\varepsilon}^{\hat{a}+\varepsilon}(\frac{n^{2H}}{2a})^{n/2}\exp(-\frac{n^{2H}}{2a}S_n)da\big].\end{aligned} \tag{2.37}$$

Since $c_1(H) > 2\,|c_2(H)|$ for each $H\in(\frac{1}{2},1)$, the eigenvalues of the matrix $(\Gamma_H)^{-1}_{(n),app}$ are positive, i.e. the matrix $(\Gamma_H)^{-1}_{(n),app}$ is positive definite and the sum S_n is positive.

By Legesgue's dominated convergence theorem, Equation (2.37) can be written as

$$\begin{aligned}
&\mathbf{P}(\hat{a}-\varepsilon<\bar{a}<\hat{a}+\varepsilon)\\
&=\frac{\left|(\Gamma_H)_{(n)}\right|^{-\frac{1}{2}}}{\mathbf{C}_{(n)}}\frac{\beta}{\pi^{n/2}}\int_{\hat{a}-\varepsilon}^{\hat{a}+\varepsilon}\mathbb{E}[(\frac{n^{2H}}{2a})^{n/2}\exp(-\frac{n^{2H}}{2a}S_n)da]\\
&\geq\frac{\left|(\Gamma_H)_{(n)}\right|^{-\frac{1}{2}}\beta}{\mathbf{C}_{(n)}(\pi\mathbb{E}(S_n))^{n/2}}\int_{\bar{a}-\varepsilon}^{\bar{a}+\varepsilon}(\frac{\mathbb{E}(S_n)n^{2H}}{2a})^{n/2}\exp(-\frac{n^{2H}}{2a}\mathbb{E}(S_n))da
\end{aligned}$$

where $\mathbf{C}_{(n)}$ is a normalization constant.

Changing the variable $t=\mathbb{E}(S_n)/2a$ and calculating the normalization constant as in the proof of Theorem 1.16, we obtain

$$\mathbf{P}(\hat{a}-\varepsilon<\bar{a}<\hat{a}+\varepsilon)\geq\frac{c}{\Gamma(n/2-1)}\int_{b_n}^{d_n}t^{n/2-2}\,e^{-t}dt=\frac{I(N)}{\Gamma(N+1)}$$

where $b_n=\dfrac{\mathbb{E}(S_n)n^{2H}}{2(\hat{a}+\varepsilon)}$, $d_n=\dfrac{\mathbb{E}(S_n)n^{2H}}{2(\hat{a}-\varepsilon)}$ and $N=\dfrac{n}{2}-2$.

Letting $t=Nu$ we get

$$I(N)=\int_{b_n/N}^{d_n/N}e^{-uN}(uN)^N\,Ndu=N^{N+1}\int_{b_n/N}^{d_n/N}e^{-N(u-\log u)}du.$$

As in the case of BM, we need to estimate the interval $[b_n/N,d_n/N]$ in the limit for $n\to\infty$. From Equation (2.35) we get in the limit for $n\to\infty$, $\forall\varepsilon<\hat{a}$

$$\begin{aligned}
\lim_{n\to\infty}\frac{b_n}{N}&=\lim_{n\to\infty}\frac{\mathbb{E}(S_n)\,n^{2H}}{2N\,(\hat{a}+\varepsilon)}\\
&=\lim_{n\to\infty}\frac{\hat{a}\,n^{2H}(n\,(1+b(H))-(n-1)l(H)^2)}{2\,n^{2H}(1+b(H)-2\,l(H)^2)(n/2-2)(\hat{a}+\varepsilon)}>\frac{1}{2}.
\end{aligned}$$

Hence $[b_n/N,d_n/N]\subset[\frac{1}{2},\infty)$.

Then, applying Lemma 1.11 and using Stirling's formula in the same way as in the proof of Theorem 1.16, we obtain

$$\begin{aligned}
\mathbf{P}(\hat{a}-\varepsilon<\bar{a}<\hat{a}+\varepsilon)&\geq\frac{I(N)}{\Gamma(N+1)}\sim\frac{N^{N+1}e^{-Nf(c)}}{N^{N+1/2}\,e^{-N}\sqrt{2\pi}}\frac{\sqrt{2\pi}}{\sqrt{Nf''(c)}}\\
&=\frac{N^{1/2}e^{-N(1-\log 1)}}{e^{-N}\sqrt{N\cdot 1}}=1,\qquad as\quad n\quad\to\infty.
\end{aligned}$$

Since a probability cannot be greater than 1 the statement is proved. ■

Summarizing, in this section we have studied the problem of estimating the amplitude parameter of FBM. The main goal was to prove that the associated discrete problem gives the correct solution in the limit. We obtained an encouraging result: if the number of realizations is large enough, the estimator $\bar{a}$ for the amplitude parameter is equal to $\hat{a}$ with probability one. Unfortunately, we have not been able to prove this result for the exact problem as was the case in BM. Since we can no more use the law of large numbers, we constructed an ergodic new process and calculated an approximation to the inverse covariance matrix in order to be able to calculate the quadratic term. Moreover, due to a technical assumption, the inverse covariance matrix could only be approximated for $\frac{1}{2} < H < 1$. Nevertheless, the computer simulations give support for believing that the result is also valid for the original problem.

2.5 The posterior distribution of the Hurst parameter

In this section, we study the behavior of the posterior distribution as a function of the Hurst parameter H.

We would like to prove a result analogous to Theorem 2.36 for H.

Consider a FBM realization with an amplitude parameter $\hat{a}$ and Hurst parameter $\hat{H}$. In this case, for simplicity, we will take the amplitude parameter as known and equal to one, so the process is:

$$Z(t) = Z_{\hat{H}}(t, \omega). \tag{2.38}$$

The relations formulated in the beginning of this chapter become in this case

1. $E(Z(t)) = 0$, for all $t \geq 0$
2. $Var(Z(t)) = |t|^{2\hat{H}}$, for all $t \geq 0$
3. $E[(Z(t+h) - Z(t))(Z(t+h) - Z(t))] = \left((\Sigma_{\hat{H}})_{(n)}\right)_{st} = \frac{1}{2}(s^{2\hat{H}} + t^{2\hat{H}} - |t-s|^{2\hat{H}})$ for all $t, s \geq 0$.

Let $\mathbf{Z}_{(n)} = (Z(t_1, \omega), ..., Z(t_n, \omega))^T$ be the vector of the observed values of the process defined above at n time instants $t_1, ..., t_n$. The conditional distribution of the process $\mathbf{Z}_{(n)}$ given that the random variable H is equal to $\hat{H}$ is

$$D_{post}(\hat{H}, \mathbf{Z}_{(n)}) = D_{pr}(\hat{H})\, D(\mathbf{Z}_{(n)}|\hat{H}). \tag{2.39}$$

As mentioned in section 2.1, we suppose there is no prior information on the value of the parameter $\hat{H}$, i.e. $D_{pr}(\hat{H}) = c$ and the posterior distribution is

$$D_{post}(\hat{H}, \mathbf{Z}_{(n)}) = \beta(2\pi)^{-n/2} \left|(\Sigma_{\hat{H}})_{(n)}\right|^{-1/2} \exp(-\frac{1}{2}\, \mathbf{Z}_{(n)}^T (\Sigma_{\hat{H}})_{(n)}^{-1} \mathbf{Z}_{(n)}), \tag{2.40}$$

where $\beta = (t_n - t_1)^{-1}$.

Next, we prove that the mass of the posterior distribution of the Hurst parameter H is asymptotically concentrated on the point $\hat{H}$ as happened for the amplitude parameter a.

Once again, due to the complexity of the covariance matrix, we consider the fGn associated to the process $Z(t)$.

Let $\mathbf{W}_{(n)} = (W_1, ..., W_n)^T$ be the realization of the stochastic process defined by

$$W_i = Z(t_i) - Z(t_{i-1})$$

where $t_i = \dfrac{i}{n}$ and with $i = 0, ..., n$.

Let $(\hat{\Gamma}_H)_{(n)}$ be the covariance matrix of the process $\mathbf{W}_{(n)}$. Then

$$(\hat{\Gamma}_H)_{(n)} = \frac{1}{n^{2H}}(\Gamma_H)_{(n)},$$

where $(\Gamma_H)_{(n)}$ is the matrix defined in (2.6), and the posterior distribution the unknown parameter H for the process $\mathbf{W}_{(n)}$ is

$$D_{post}(H, \mathbf{W}_{(n)}) = \beta\,(\frac{2\pi}{n^{2H}})^{-n/2}\left|(\Gamma_H)_{(n)}\right|^{-1/2}\exp\big(-\frac{n^{2H}}{2}\mathbf{W}_{(n)}^T(\Gamma_H)_{(n)}^{-1}\mathbf{W}_{(n)}\big).$$

Since the matrix $(\Gamma_H)_{(n)}$ is exactly the one defined in Equation (2.6), we can proceed in the same way than in section 2.2 to calculate its inverse.

The measurement vector $\mathbf{W}_{(n)}$ is fixed and does not depend on H. On the other hand, the determinant of the covariance matrix depends on H and has to be calculated explicitly. Again, due to the complexity of the covariance matrix, it is not possible to do this analytically. We use an approximate inverse covariance matrix.

It is clear that for the determinant of the covariance matrix satisfies

$$\left|(\Gamma_H)_{(n)}\right| = \frac{1}{\left|(\Gamma_H)_{(n)}^{-1}\right|}.$$

Since the matrix $(\Gamma_H)_{(n),app}^{-1}$ is a band matrix, we need the following lemma to calculate its determinant.

Lemma 2.41. *Suppose $f, g \in C^1[0,1]$ and suppose that $f = |f| \geq 2\,|g|$. Let $M(n,H)$ be an $n \times n$ symmetric tridiagonal Toeplitz matrix such that $M(n,H)_{11} = f(H)$ and $M(n,H)_{12} = g(H)$. Denote by $h(H) = \sqrt{f(H)^2 - 4g(H)^2}$ and by $j(H) = f(H) + h(H)$. Then*

$$\det(M(n,H)) \sim \frac{1}{h(H)}\Big(\frac{j(H)}{2}\Big)^{n+1} \tag{2.42}$$

and if $j'(H) \neq 0$ then

$$\partial_H\big(\det(M(n,H))\big) \sim \frac{nj'(H)}{2h(H)}\Big(\frac{j(H)}{2}\Big)^{n}. \tag{2.43}$$

Proof. For simplicity, we will consider the matrix

$$N(n,H) = \begin{pmatrix} 1 & c & 0 & \cdots & \cdots & \cdots \\ c & 1 & c & 0 & \cdots & \cdots \\ 0 & c & 1 & c & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & c & 1 & c \\ \cdots & \cdots & \cdots & 0 & c & 1 \end{pmatrix},$$

where $c = g(H)/f(H)$ and $|c| < \frac{1}{2}$.

Then

$$\det M(n,H) = f(H)^n \det N(n,H).$$

We note that the aymptotic relations (2.42) and (2.43) hold if we show that

$$D(n) = \det N(n,H) \sim \frac{1}{a}\Big(\frac{1+a}{2}\Big)^{n+1} = \lambda(n) \qquad \text{with} \qquad a = \sqrt{1-4c^2}$$

and that

$$\partial_H\big(\det(M(n,H))\big) \sim \partial_H\big(\lambda(n)\big).$$

To this aim, we define $R(n)$ by

$$D(n) = \lambda(n)R(n)$$

and show that

$$R(n) \underset{n\to\infty}{\longrightarrow} 1$$

and

$$\partial_H\big(\lambda(n)\big) \underset{n\to\infty}{\longrightarrow} 0.$$

In the following, we use the Fibonacci-type recursion equation

$$D(n) = D(n-1) - c^2 D(n-2) \qquad \forall n \geq 2, \tag{2.44}$$

and define $D(0) = D(1) = 1$ so that Equation (2.44) is true for $n = 2$.

For $\Delta(n) = R(n) - R(n-1)$ we can write the recursion equation

$$\Delta(n) = c^2 \frac{\lambda(n-2)}{\lambda(n)} \Delta(n-1). \tag{2.45}$$

Defining

$$\rho = c^2 \frac{\lambda(n-2)}{\lambda(n)} = \frac{1-a}{1+a} < 1,$$

Equation (2.45) can be rewritten as

$$\Delta(n) = \rho\Delta(n-1) = \cdots = \rho^{n-1}\Delta(1).$$

This is equivalent with

$$\begin{aligned} R(n) - R(n-1) = &\quad \rho^{n-1}\Delta(1) \\ R(n-1) - R(n-2) = &\quad \rho^{n-2}\Delta(1) \\ \cdots & \\ R(1) - R(0) = &\quad \Delta(1). \end{aligned}$$

This means that

$$R(n) - R(0) = \Delta(1) \sum_{i=0}^{n-1} \rho^i,$$

and hence

$$R(n) = R(0) + \frac{R(1) - R(0)}{1-\rho}\,(1-\rho^n).$$

Since

$$R(0) = \frac{2a}{1+a} = 1 - \rho \quad \text{and} \quad R(1) = \frac{4a}{(1+a)^2} = \frac{2R(0)}{1+a},$$

it follows

$$R(1) - R(0) = R(0)\rho.$$

Hence, since

$$R(n) = 1 - \rho^{n+1}$$

it easily seen that

$$R(n) \underset{n\to\infty}{\longrightarrow} 1$$

and that

$$\partial_H R(n) = \partial_H \Big(1 - \rho^{n+1}\Big) \underset{n\to\infty}{\longrightarrow} 0.$$

This proves the theorem. ■

In particular, if we consider

$$f(H) = c_1(H)$$

and

$$g(H) = c_2(H)$$

with $c_1(H)$ and $c_2(H)$ as defined in Equations (2.28) and (2.29), it follows that the matrix $(M(n,H))$ of Lemma 2.41 coincides with the approximate inverse covariance matrix $(\Gamma_H)^{-1}_{(n),app}$. It can be easily shown that $c_1(H) \geq 2\,|c_2(H)|$ for each $H \in (0,1)$, and we can use the result of Lemma 2.41 to calculate explicitly the posterior distribution.

In the rest of this section, we will consider the following problem. Given the posterior distribution

$$D^{app}_{post}(H, \mathbf{W}_{(n)}) = \beta\,(\frac{2\pi}{n^{2H}})^{-n/2}\left|(\Gamma_H)_{(n),app}\right|^{-1/2} \exp(-\frac{n^{2H}}{2}\mathbf{W}^T_{(n)}(\Gamma_H)^{-1}_{(n),app}\mathbf{W}_{(n)}), \tag{2.46}$$

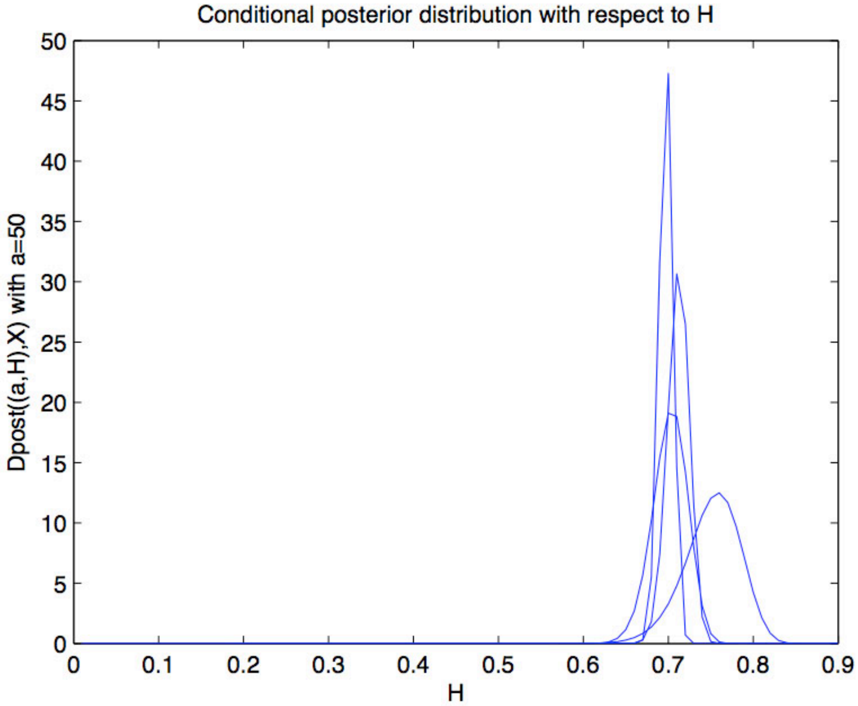

Figure 2.2: Posterior distributions of a realization of the same process with a different number of measurements.

where $(\Gamma_H)^{-1}_{(n),app}$ as in (2.6), prove that

$$\lim_{n\to\infty} \mathbf{P}(\hat{H} - \varepsilon < \bar{H} < \hat{H} + \varepsilon) = 1,$$

where $\bar{H}$ is the estimator of the Hurst parameter.

Using the notation of section 2.3, we can rewrite the posterior distribution as

$$D^{app}_{post}(H, \mathbf{W}_{(n)}) = \beta(\frac{2\pi}{n^{2H}})^{-n/2} \frac{1}{h(H)} \Big(\frac{j(H)}{2}\Big)^{n+1} \exp\big(-\frac{n^{2H}}{2}(c_1 Q_n + 2c_2 V_n)\big), \quad (2.47)$$

where

$$Q_n = \sum_{i=0}^{n} W_i^2 \quad (2.48)$$

and

$$V_n = \sum_{i=0}^{n-1} W_i \cdot W_{i-1} \quad (2.49)$$

and $j(H)$ and $h(H)$ as defined in Lemma 2.41.

To proceed in the same manner as in the case of the amplitude parameter, we should integrate Equation (2.47) with respect to H. In this way we would get a result analogous to Theorem 2.36.

Unfortunately, we have not been able to evaluate the integral of the posterior distribution (2.47) asymptotically with respect to H. As shown in Figure 2.2, the numerical simulations show that the mass of the posterior distribution tends to concentrate on the true value of the Hurst parameter when the number of observations increases.

Even if weaker then the original proposition, it would be a good result from the applied point of view to prove that the posterior distribution $D^{app}_{post}(H, \mathbf{W}_{(n)})$ attains its maximum for $H = \hat{H}$ as n tends to infinity. This would guarantee that the maximum of the posterior distribution gives the true value for the estimator based on the approximate inverse covariance matrix. This would also prove that we can solve the associated discrete problem and get the asymptotically correct value of H by the statistical inversion method.

In order to calculate the maximum, we have to calculate the derivative of the posterior distribution in (2.47). We have succeeded in proving that the derivative of the posterior distribution is positive for $H < \hat{H}$ and negative for $H > \hat{H}$ when n is large enough. From this, we can deduce that the posterior distribution is an increasing function for $H < \hat{H}$ and a decreasing one for $H > \hat{H}$. Hence the posterior distribution attains its maximum value at $H = \hat{H}$.

In the following theorem, we study the sign of the derivative of the posterior distribution.

Theorem 2.50. *Let*

$$D'(H) = \frac{d}{dH} D^{app}_{post}(H, \mathbf{W}_{(n)}). \tag{2.51}$$

Then for n large enough, the function $D'(H)$ is positive for $H < \hat{H}$ and negative for $H > \hat{H}$ with $H \in (\frac{1}{2}, 1)$ and $n > m$.

Proof. We start by calculating the derivative of the posterior distribution in (2.47)with respect to H. We obtain

$$\begin{aligned}\frac{d}{dH} D^{app}_{post}(H, \mathbf{W}_{(n)}) &= \beta (2\pi)^{-n/2}\, n^{nH} \frac{1}{h(H)} \Big(\frac{j(H)}{2}\Big)^{n+1} \exp\big(-\frac{n^{2H}}{2}(c_1 Q_n + 2c_2 V_n)\big) \\ &\cdot \big[n \log n + \frac{n j'(H)}{j(H)} - n^{2H} \log n (c_1 Q_n + 2c_2 V_n) - \frac{n^{2H}}{2}(\frac{d}{dH} c_1\, Q_n + 2\frac{d}{dH} c_2 V_n)\big],\end{aligned} \tag{2.52}$$

where $c_1(H)$ and $c_2(H)$ are defined in (2.28) and (2.29) and $j(H)$ and $h(H)$ in Lemma 2.41.

The elements $c_1(H)$ and $c_2(H)$ depend on H, but not on n. Using the same argument as in section 2.3, it follows in the limit for $n \to \infty$ from the Corollary 2.34 that

$$\frac{Q_n}{n} \sim Q(n) \qquad \text{and} \qquad \frac{V_n}{n-1} \sim V(n)$$

with

$$\mathbb{E}(Q(n)) = \mathbb{E}((W_0)^2) = \mathbb{V}ar(W_0) = \frac{1}{n^{2\hat{H}}}$$

and

$$\mathbb{E}(V(n)) = \mathbb{E}(W_i W_{i-1}) = \mathbb{C}ov(W_i, W_{i-1}) = \frac{1}{n^{2\hat{H}}}(2^{2\hat{H}-1} - 1).$$

Since Q_n and V_n do not depend on H, we can deduce the following by studying each term of (2.52):

1. The term
$$\beta(2\pi)^{-n/2}\, n^{nH} \frac{1}{h(H)} \Big(\frac{j(H)}{2}\Big)^{n+1} \exp(-\frac{n^{2H}}{2}(c_1 Q_n + 2c_2 V_n))$$
is always positive $\forall H \in (\frac{1}{2}, 1)$.
2. The term
$$\big[n \log n + \frac{nj'(H)}{j(H)} - n^{2H} \log n(c_1 Q_n + 2c_2 V_n) - \frac{n^{2H}}{2}(\frac{d}{dH}c_1 Q_n + 2\frac{d}{dH}c_2 V_n)\big], \tag{2.53}$$
is the one which determines the sign in Equation (2.52).

For large enough n, there are only two important terms when studying the sign in (2.53). The first one is $n \log n$ and the second one, as n tends to infinity, is

$$\begin{aligned}
&n^{2H} \log n\big(c_1(H) Q_n + 2c_2(H) V_n\big) \\
&\sim n^{2H} \log n\big(nc_1(H)Q(n) + 2(n-1)c_2(H)V(n)\big) \\
&= n^{2H} \log n\big(nc_1(H)n^{-2\hat{H}}Q^{\text{inf}} + 2(n-1)c_2(H)n^{-2\hat{H}}V^{\text{inf}}\big) \\
&= n^{2(H-\hat{H})} \log n\big(nc_1(H)Q^{\text{inf}} + 2(n-1)c_2(H)V^{\text{inf}}\big),
\end{aligned}$$

where Q^{inf} and V^{inf} are analogous to T^{inf} and U^{inf}, defined in Equations (2.32) and (2.33), do not depend on n.

In particular, if $H < \hat{H}$ then $(2(H - \hat{H}) + 1) < 1$, the dominating term in (2.53) is $n \log n$ and the derivative in (2.52) is positive.

On the other hand, if $H > \hat{H}$ then $(2(H - \hat{H}) + 1) > 1$, hence the dominating term in Equation (2.53) is

$$\frac{n^{2(H-\hat{H})}}{2} \log n\big(nc_1(H) + 2(n-1)c_2(H)(2^{2\hat{H}-1} - 1)\big).$$

Since $(nc_1(H) + 2(n-1)c_2(H)(2^{2\hat{H}-1} - 1))$ is positive for

$$n > \frac{2c_2(H)(2^{2\hat{H}-1} - 1)}{c_1(H) + 2c_2(H)(2^{2\hat{H}-1} - 1)}$$

and for $\frac{1}{2} < H < 1$, the derivative in Equation (2.52) is negative. ■

Summarizing, we have studied the problem of retrieving the Hurst parameters from the realization of FBM. We considered the approximate problem in order to compute the posterior distribution. We studied the sign of the derivative of the function $D_{post}^{app}(H, \mathbf{W}_{(n)})$ for $\frac{1}{2} < H < 1$ and proved that the solution obtained by applying the statistical inversion method coincides with the true value of the estimator when the number of measurements is large enough.

Chapter 3

Retrieving the underlying parameters of the sum of two FBMs

Physical measurements as a function of time contain trend and noise. The main problem is to succeed in cleaning up the data from the noise. Traditionally, when modelling physical phenomena, the noise is assumed to be Gaussian.

This characteristic is preserved when formulating the following problem.

We model both the trend and the noise as realizations of FBM. The measurement vector $(X(t_1), ..., X(t_n))$ is supposed to be the sum of these two realizations. In particular, $X(t_i) = X^t(t_i) + X^n(t_i)$, where $X^t(t_i)$ is the realization modelling the trend with underlying parameters a^t and H^t, and $X^n(t_i)$ the realization modelling the noise with parameters a^n and H^n. Our goal is to estimate the underlying parameters so that we can retrieve the trend.

The problem is to identify the signals, i.e. to retrieve the amplitudes a^t, a^n and Hurst parameters H^t and H^n.

Intuitively, this ought to be possible at least for the dominating part of the measured signal, i.e. the one with the smaller Hurst parameter H^n. In the following, in order to get a better understanding of the problem, we use different scenarios.

There are obviously cases where it is not possible to recover all four parameters a^t, a^n, H^t and H^n. For example, if we consider the case $H^t = H^n$, we have two realizations of the same process with different amplitude parameters. We show in that in this case, one can only hope to recover the sum $a^t + a^n$ of the corresponding amplitude parameters.

After the case $H^t = H^n$ we can suppose that $H^t > H^n$ without loss of generality. Note that a sample path of FBM gets smoother as the Hurst parameter H increases. In practice, the realization of a FBM with a larger Hurst parameter H^t models well the trend contained in the measurement data. It is therefore reasonable to assume that the trend amplitude a^t is larger than the noise amplitude a^n, since the noise is usually much smaller than the trend. For this reason, we assume that $a^t > a^n$.

3.1 The case $H^t = H^n$

The first case is the sum of two realizations of FBM both with the same Hurst parameter. We show that we can recover the sum $a^t + a^n$ asymptotically.

To this aim, we consider two measurement vectors $\mathbf{X}^A_{(n)}$ and $\mathbf{X}^B_{(n)}$. Each of them is the sum of two FBMs. Starting with the assumption that the posterior distributions of the measurement vectors are the same, we study whether the amplitude parameters are the same.

Consider two stochastic processes $\mathbf{X}^A_{(n)} = (X^A(t_1), ..., X^A(t_n))$ and $\mathbf{X}^B_{(n)} = (X^B(t_1), ..., X^B(t_n))$ defined as:

$$\begin{aligned} X^A(t_i) &= \sqrt{a^t_A} Z_{H^t_A}(t_i) + \sqrt{a^n_A} Z_{H^n_A}(t_i) \\ X^B(t_i) &= \sqrt{a^t_B} Z_{H^t_B}(t_i) + \sqrt{a^n_B} Z_{H^n_B}(t_i). \end{aligned} \tag{3.1}$$

Since we are going to study the case of two realizations with the same Hurst parameters, we suppose $H^t_A = H^n_A = H_A$ and $H^t_B = H^n_B = H_B$. We can then rewrite the Equations (3.1) as:

$$\begin{aligned} X^A(t_i) &= \sqrt{a^t_A} Z_{H_A}(t_i, \omega_1) + \sqrt{a^n_A} Z_{H_A}(t_i, \omega_2) \\ X^B(t_i) &= \sqrt{a^t_B} Z_{H_B}(t_i, \theta_1) + \sqrt{a^n_B} Z_{H_B}(t_i, \theta_2), \end{aligned} \tag{3.2}$$

where $\omega_i, \theta_i \in \Omega$ for $i = 1, 2$. Note that the two components of each sum can be different realizations of the same FBM.

In order to deal with the two covariance matrices and their inverses, we consider the associated fGn and define $\mathbf{Y}^A_{(n)} = (Y^A(t_1), ..., Y^A(t_n))$, and $\mathbf{Y}^B_{(n)} = (Y^B(t_1), ..., Y^B(t_n))$ as

$$\begin{aligned} Y^A(t_i) &= X^A(t_i) - X^A(t_{i-1}) \\ Y^B(t_i) &= X^B(t_i) - X^B(t_{i-1}). \end{aligned} \tag{3.3}$$

The covariance matrices $(\Lambda_k)_{(n)}$ with $k = A, B$ for the processes defined in (3.3) will be of the type:

$$(\Lambda_k)_{(n)} = \frac{n^{2H_k}}{a^t_k + a^n_k} (\Gamma_{H_k})_{(n)},$$

where $(\Gamma_{H_k})_{(n)}$ is the covariance matrix as defined in Equation (2.6).

As in section 2.2, we consider the approximate problem in order to be able to calculate the posterior distribution explicitly.

Using the notation in section 2.3, the approximate posterior distributions for the processes $\mathbf{Y}^k_{(n)}$ with $k = A, B$ as a function of the unknown parameters H, a^t and a^n with $\frac{1}{2} < H < 1$ are:

$$\begin{aligned}
&D^{app}_{post}(\mathbf{Y}^k_{(n)}, H, a^t, a^n) = \\
&\frac{\beta}{(2\pi)^{\frac{1}{2}}} n^{nH} (\frac{a^t + a^n}{n^{2H}})^{\frac{n}{2}} \left|(\Gamma_H)_{(n),app}\right|^{-\frac{1}{2}} \exp\big(- \frac{n^{2H}}{2(a^t + a^n)} (\mathbf{Y}^k_{(n)})^T (\Gamma^{-1}_H)_{(n),app} \mathbf{Y}^k_{(n)}\big) = \\
&\frac{\beta}{(2\pi)^{\frac{1}{2}}} n^{nH} (\frac{a^t + a^n}{n^{2H}})^{\frac{n}{2}} \frac{1}{h(H)} \Big(\frac{j(H)}{2}\Big)^{n+1} \exp\big(- \frac{n^{2H}}{2(a^t + a^n)} (c_1(H) T^k_n + 2c_2(H) U^k_n)\big)
\end{aligned} \tag{3.4}$$

where $T^k_n = \sum_{i=0}^n (Y^k(t_i))^2$ and $U^k_n = \sum_{i=0}^n Y^k(t_i) Y^k(t_{i+1})$ do not depend on H, a^t or a^n.

The problem of estimating the underlying parameters of the sum of two FBMs with the same Hurst parameter leads to the following theorem:

Theorem 3.5. *Consider the stochastic processes $\mathbf{Y}^A_{(n)}$ and $\mathbf{Y}^B_{(n)}$ defined in (3.3) with $\frac{1}{2} < H_A, H_B < 1$. Then*

$$D^{app}_{post}(\mathbf{Y}^A_{(n)}, H, a^t, a^n) = D^{app}_{post}(\mathbf{Y}^B_{(n)}, H, a^t, a^n)$$

as $n \to \infty$ if and only if

$$H_A = H_B \quad \text{and} \quad a^t_A + a^n_A = a^t_B + a^n_B.$$

Proof. From Equation 3.4 it follows that

$$D^{app}_{post}(\mathbf{Y}^A_{(n)}, H, a^t, a^n) = D^{app}_{post}(\mathbf{Y}^B_{(n)}, H, a^t, a^n)$$

if and only if

$$\begin{aligned}
&(\frac{a^t + a^n}{n^{2H}})^{n/2} \frac{1}{h(H)} \Big(\frac{j(H)}{2}\Big)^{n+1} \exp(-\frac{n^{2H}}{2(a^t + a^n)} (c_1(H) T^A_n + 2c_2(H) U^A_n)) = \\
&\quad (\frac{a^t + a^n}{n^{2H}})^{n/2} \frac{1}{h(H)} \Big(\frac{j(H)}{2}\Big)^{n+1} \exp(-\frac{n^{2H}}{2(a^t + a^n)} (c_1(H)) T^B_n + 2c_2(H) U^B_n)),
\end{aligned}$$

where $c_1(H)$ and $c_2(H)$ are the elements of the approximate inverse covariance matrix as defined in (2.28) and (2.29).

This is equivalent to

$$\exp\big(-\frac{n^{2H}}{2(a^t + a^n)} (c_1(H) T^A_n + 2c_2(H) U^A_n)\big) = \exp\big(-\frac{n^{2H}}{2(a^t + a^n)} (c_1(H) T^B_n + 2c_2(H) U^B_n)\big),$$

so it is enough to prove that

$$c_1(H)(T^A_n - T^B_n) = 2c_2(H)(U^B_n - U^A_n). \tag{3.6}$$

Applying Corollary 2.34 to the sequences T_n^A, U_n^A, T_n^B and U_n^B as in section 2.3, it follows for $k = A, B$ in the limit $n \to \infty$

$$\frac{T_n^k}{n} \sim \frac{a_k^t + a_k^n}{n^{2H_k}}$$

and

$$\frac{U_n^k}{n-1} \sim \frac{a_k^t + a_k^n}{n^{2H_k}}(2^{2H_k-1} - 1).$$

Equation (3.6) implies

$$\begin{aligned} &nc_1(H)(\frac{a_A^t + a_A^n}{n^{2H_A}} - \frac{a_B^t + a_B^n}{n^{2H_B}}) \\ \sim &2(n-1)c_2(H)\big(\frac{a_B^t + a_B^n}{n^{2H_B}}(2^{2H_B-1} - 1) - \frac{a_A^t + a_A^n}{n^{2H_A}}(2^{2H_A-1} - 1)\big). \end{aligned}$$

This is equivalent to

$$\begin{aligned} &\frac{a_A^t + a_A^n}{n^{2H_A}}\big(nc_1(H) + 2(n-1)c_2(H)(2^{2H_A-1} - 1)\big) \\ \sim &\frac{a_B^t + a_B^n}{n^{2H_B}}\big(nc_1(H) + 2(n-1)c_2(H)(2^{2H_B-1} - 1)\big). \end{aligned}$$

This equation is obviously satisfied if and only if $H_A = H_B$. On the other hand, this condition implies that

$$a_A^t + a_A^n = a_B^t + a_B^n.$$

■

This proves that we are not able to recover the values of the amplitude parameters of the two FBM realizations from their sum when the Hurst parameters are equal. In this case, there is no unique solution.

3.2 Uniqueness of the amplitude solution

In this section, we study the posterior distributions of the sum of two FBM realizations with unequal Hurst parameters. The goal is to prove the existence and uniqueness of the solutions in the case the Hurst parameters are known in retrieving the amplitude parameters a^t and a^n.

Even if the computer simulations in some cases support the statement that the solution is uniquely determined for both a^t and a^n, we have only succeeded to prove part of this result analytically. In the following theorem, we prove that the amplitude parameter of the realization with the *smaller* Hurst parameter can be retrieved uniquely.

In this section, we suppose that the Hurst parameters are known. For simplicity let $H_1^t = H_2^t = H^t$ and $H_1^n = H_2^n = H^n$. Since by Theorem 3.5 the condition $H^t \neq H^n$ is necessary for the uniqueness of the solution, we only consider the case $H^n < H^t$.

Consider two stochastic processes $\mathbf{X}_{(n)}^{A} = (X^A(t_1), ..., X^A(t_n))$, and $\mathbf{X}_{(n)}^{B} = (X^B(t_1), ..., X^B(t_n))$ as defined in Equation (3.1). Since the Hurst parameters are known, Equation (3.1) can be rewritten as

$$\begin{aligned} X^A(t_i) &= \sqrt{a_A^t} Z_{H^t}(t_i, \omega_1) + \sqrt{a_A^n} Z_{H^n}(t_i, \omega_2) \\ X^B(t_i) &= \sqrt{a_B^t} Z_{H^t}(t_i, \theta_1) + \sqrt{a_B^n} Z_{H^n}(t_i, \theta_2), \end{aligned} \tag{3.7}$$

where $\omega_i, \theta_i \in \Omega$ for $i = 1, 2$. Note that even if the FBM realizations have the same Hurst parameters, they can be two different realizations of the same process.

In order to simplify the explicit calculation we define the associated fGn. Define $\mathbf{Y}_{(n)}^{A} = (Y^A(t_1), ..., Y^A(t_n))$, and $\mathbf{Y}_{(n)}^{B} = (Y^B(t_1), ..., Y^B(t_n))$ as follows

$$\begin{aligned} Y^A(t_i) &= X^A(t_i) - X^A(t_{i-1}) \\ Y^B(t_i) &= X^B(t_i) - X^B(t_{i-1}). \end{aligned} \tag{3.8}$$

The covariance matrices $(\Lambda^k)_{(n)}$ with $k = A, B$ are then

$$(\Lambda^k)_{(n)} = \frac{a_k^t}{n^{2H^t}} (\Gamma_{H^t})_{(n)} + \frac{a_k^n}{n^{2H^n}} (\Gamma_{H^n})_{(n)}$$

where $(\Gamma_H)_{(n)}$ is the covariance matrix defined in Equation (2.6).

Once more, since we need the inverse covariance matrix explicitly, we consider the approximate problem as in section 2.2. Using the notation in section 2.3, the posterior distribution for the processes $\mathbf{Y}_{(n)}^{k}$ given $a^t = a_k^t$ and $a^n = a_k^n$ with $k = A, B$ is

$$D_{post}^{app}(\mathbf{Y}_{(n)}^{k}, a_k^t, a_k^n) = \frac{\beta}{(2\pi)^{n/2}} \left|(\Lambda^k)_{(n),app}\right|^{-\frac{1}{2}} \exp(-\frac{1}{2}(\mathbf{Y}_{(n)}^{k})^T (\Lambda^k)_{(n),app}^{-1} \mathbf{Y}_{(n)}^{k}). \tag{3.9}$$

In this case, we cannot take the underlying parameters out of the *approximate* covariance matrix to calculate the inverse. We use the again the method presented in section 2.2 to calculate the elements $c_1(a^t, a^n)$ and $c_2(a^t, a^n)$ of the full inverse covariance matrix.

Using the notation in section 2.2, for the processes $\mathbf{Y}_{(n)}^{k}$ $k = A, B$ we have:

$$\begin{aligned} [(\Lambda^k)_{(n)}]_{3\times 3} =& \frac{a_k^t}{n^{2H^t}} [(\Gamma_{H^t})_{(n)}]_{3\times 3} + \frac{a_k^n}{n^{2H^n}} [(\Gamma_{H^n})_{(n)}]_{3\times 3} \\ =& \frac{a_k^t}{n^{2H^t}} \begin{pmatrix} 1 & l(H^t) & b(H^t) \\ l(H^t) & 1 & l(H^t) \\ b(H^t) & l(H^t) & 1 \end{pmatrix} + \frac{a_k^n}{n^{2H^n}} \begin{pmatrix} 1 & l(H^n) & b(H^n) \\ l(H^n) & 1 & l(H^n) \\ b(H^n) & l(H^n) & 1 \end{pmatrix} \\ =& \begin{pmatrix} f(a_k^t, a_k^n) & F(a_k^t, a_k^n) & g(a_k^t, a_k^n) \\ F(a_k^t, a_k^n) & f(a_k^t, a_k^n) & F(a_k^t, a_k^n) \\ g(a_k^t, a_k^n) & F(a_k^t, a_k^n) & f(a_k^t, a_k^n) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned}
f(a_k^t, a_k^n) &= \frac{a_k^t}{n^{2H^t}} + \frac{a_k^n}{n^{2H^n}} \\
F(a_k^t, a_k^n) &= \frac{a_k^t}{n^{2H^t}} l(H^t) + \frac{a_k^n}{n^{2H^n}} l(H^n) \\
g(a_k^t, a_k^n) &= \frac{a_k^t}{n^{2H^t}} b(H^t) + \frac{a_k^n}{n^{2H^n}} b(H^n),
\end{aligned}$$

where $b(H)$ and $l(H)$ are defined as in (2.25) and (2.24) .

The determinant of the matrices $[(\Lambda^k)_{(n)}]_{3\times 3}$ for $k = A, B$ will be

$$\begin{aligned}
\left|[(\Lambda^k)_{(n)}]_{3\times 3}\right| =& f(a_k^t, a_k^n)\big[f(a_k^t, a_k^n)^2 - g(a_k^t, a_k^n)^2 b(H^n)\big] \\
&-2F(a_k^t, a_k^n)^2 b(H^n)\big[f(a_k^t, a_k^n) - g(a_k^t, a_k^n) b(H^n)\big].
\end{aligned}$$

For $\frac{1}{2} < H^t, H^n < 1$ we can invert the covariance matrix. The elements of the approximate inverse covariance matrix will be

$$c_1(a_k^t, a_k^n) = \frac{f(a_k^t, a_k^n) + g(a_k^t, a_k^n)}{f(a_k^t, a_k^n)[f(a_k^t, a_k^n) + g(a_k^t, a_k^n)] + 2F(a_k^t, a_k^n)}$$

and

$$c_2(a_k^t, a_k^n) = -\frac{F(a_k^t, a_k^n)}{f(a_k^t, a_k^n)[f(a_k^t, a_k^n) + g(a_k^t, a_k^n)] + 2F(a_k^t, a_k^n)}.$$

Note that the elements c_1 and c_2 depend on the amplitude parameters a^t and a^n.

Using the notation in section 2.3 for the stochastic processes $\mathbf{Y}^k_{(n)}$ with $k = A, B$, it follows that

$$\mathbf{Y}^k_{(n)}(\Lambda^k)^{-1}_{(n)}(\mathbf{Y}^k_{(n)})^T = c_1(a_k^t, a_k^n)T_n^k + 2c_2(a_k^t, a_k^n)U_n^k.$$

Applying Corollary 2.34 to the sequences T_n^k and U_n^k in the limit $n \to \infty$, we get

$$\frac{T_n^k}{n} \sim f(a_k^t, a_k^n)$$

and

$$\frac{U_n^k}{n-1} \sim F(a_k^t, a_k^n).$$

Hence the posterior distribution can be estimated in the limit $n \to \infty$ as

$$\begin{aligned}
&\lim_{n\to\infty} D^{app}_{post}(\mathbf{Y}^k_{(n)}, a^t, a^n) = \\
&\lim_{n\to\infty} \beta(2\pi)^{-\frac{n}{2}} \left|(\Lambda^k)_{(n),app}\right|^{-\frac{1}{2}} \exp(-\frac{n}{2}(c_1(a_k^t, a_k^n)f(a_k^t, a_k^n) + 2c_2(a_k^t, a_k^n)F(a_k^t, a_k^n))).
\end{aligned} \tag{3.10}$$

The problem of estimating the amplitude parameters a^t and a^n of the sum of two FBM with different Hurst parameters, leads to the following theorem:

Theorem 3.11. *Consider the stochastic processes* $\mathbf{Y}^A_{(n)}$ *and* $\mathbf{Y}^B_{(n)}$ *defined in (3.8) with* $\frac{1}{2} < H^t, H^n < 1$. *Then*

$$D^{app}_{post}(\mathbf{Y}^A_{(n)}, a^t, a^n) = D^{app}_{post}(\mathbf{Y}^B_{(n)}, a^t, a^n)$$

as $n \to \infty$ *implies*

$$a^n_A = a^n_B.$$

Proof. Starting from (3.10) we can easily verify that in the limit $n \to \infty$

$$D^{app}_{post}(\mathbf{Y}^A_{(n)}, a^t, a^n) = D^{app}_{post}(\mathbf{Y}^B_{(n)}, a^t, a^n)$$

if and only if

$$\lim_{n\to\infty} \frac{\beta}{(2\pi)^{n/2}} \left|(\Lambda^k)_{(n),app}\right|^{-\frac{1}{2}} \exp\big(-\frac{n}{2}(c_1(a^t,a^n)f(a^t_A,a^n_A) + 2c_2(a^t,a^n)F(a^t_A,a^n_A))\big) =$$
$$\lim_{n\to\infty} \frac{\beta}{(2\pi)^{n/2}} \left|(\Lambda^k)_{(n),app}\right|^{-\frac{1}{2}} \exp\big(-\frac{n}{2}(c_1(a^t,a^n)f(a^t_B,a^n_B + 2c_2(a^t,a^n)F(a^t_B,a^n_B))\big).$$

We recall that the sums T^k_n and U^k_n with $k = A, B$ do not depend on the unknown parameters a^t and a^n. Therefore, the posterior distributions of the unknown amplitude parameters, i.e. the elements c_1 and c_2, depend on a^t and a^n. This is equivalent to

$$\begin{aligned}&\lim_{n\to\infty} c_1(a^t,a^n)\big[f(a^t_A,a^n_A) - f(a^t_B,a^n_B)\big] =\\ &\lim_{n\to\infty} 2c_2(a^t,a^n)\big[F(a^t_B,a^n_B) - F(a^t_A,a^n_B)\big],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&\lim_{n\to\infty} \big[f(a^t,a^n) + g(a^t,a^n))\big]\big[f(a^t_A,a^n_A) - f(a^t_B,a^n_B)\big] =\\ &\lim_{n\to\infty} -2F(a^t,a^n)\big[F(a^t_B,a^n_B) - F(a^t_A,a^n_A)\big].\end{aligned} \tag{3.12}$$

Since we have supposed that $H^t > H^n$, the dominating terms in the limit $n \to \infty$ will be those of order $1/n^{4H^n}$. Hence, if we consider only the dominating terms, Equation (3.12) can be written as

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n^{4H^n}}(a^n a^n_A + b(H^n)a^n a^n_A - a^n a^n_B - b(H^n)a^n a^n_B) =\\ &\lim_{n\to\infty} \frac{-2}{n^{4H^n}}(l(H^n)^2 a^n a^n_B - l(H^n)^2 a^n a^n_B)\end{aligned}$$

where

$$b(H^n) = \frac{1}{2}(3^{2H^n} - 2^{2H^n} + 1)$$

and

$$l(H^n) = (2^{2H^n-1} - 1).$$

This is true if and only if

$$a^n(a_A^n - a_B^n)(1 + b(H^n)) = 2l(H^n)^2 a^n(a_B^n - a_A^n).$$

Thereore, we must have

$$(a_A^n - a_B^n)\big(1 + b(H^n) + 2l(H^n)^2\big) = 0.$$

Since $1 + b(H^n) + 2l(H^n)^2) > 0$ for all $\frac{1}{2} < H^n < 1$ the condition is satisfied if and only if $a_A^n = a_B^n$. This proves the theorem. ■

We considered the problem of retrieving the amplitude parameters from the vector of the sum of two FBMs with $H^t > H^n$. Our initial goal was to prove the uniqueness of the solution with respect to both amplitude parameters, a^t and a^n. Unfortunately, the asymptotic behavior only gives the uniqueness to the amplitude parameter of the realization with the smaller Hurst parameter, i.e. the one representing the noise.

However, the simulations, lead us to believe that also in the general case, both a^t and a^n can be determined by discrete measurements. This remains an open problem.

3.3 The uniqueness of the H solution

The aim of this section is to prove an analogue to Theorem 3.11 for the Hurst parameters. In particular, we will consider the sum of two FBM realizations with different Hurst parameters H^t and H^n and assume the amplitude parameters are known and by definition, always positive. For simplicity, we assume

$$a_A^t = a_B^t = a_A^n = a_B^n = 1.$$

This will simplify the calculations. Moreover, we assume that $H^t > H^n$ as in Theorem 3.11.

Consider two stochastic processes $\mathbf{X}_{(n)}^A = (X^A(t_1), ..., X^A(t_n))$, and $\mathbf{X}_{(n)}^B = (X^B(t_1), ..., X^B(t_n))$ such that

$$\begin{aligned} X^A(t_i) &= Z_{H_A^t}(t_i) + Z_{H_A^n}(t_i) \\ X^B(t_i) &= Z_{H_B^t}(t_i) + \ Z_{H_B^n}(t_i) \end{aligned} \tag{3.13}$$

and define $\mathbf{Y}_{(n)}^A = (Y^A(t_1), ..., Y^A(t_n))$, and $\mathbf{Y}_{(n)}^B = (Y^B(t_1), ..., Y^B(t_n))$ as follows

$$\begin{aligned} Y^A(t_i) &= X^A(t_i) - X^A(t_{i-1}) \\ Y^B(t_i) &= X^B(t_i) - X^B(t_{i-1}). \end{aligned} \tag{3.14}$$

Then the covariance matrices $(\Lambda^k)_{(n)}$ for $k = A, B$ of these two processes will be

$$(\Lambda^k)_{(n)} = \frac{1}{n^{2H_k^t}}(\Gamma_{H_k^t})_{(n)} + \frac{1}{n^{2H_k^n}}(\Gamma_{H_k^n})_{(n)}$$

where $(\Gamma_H)_{(n)})$ is the covariance matrix defined in Equation (2.6).

In order to write the posterior distribution explicitly we proceed as in section 2.2. The posterior distribution of the processes $\mathbf{Y}^k_{(n)}$ for $k = A, B$ will be like Equation (3.9) with $\frac{1}{2} < H^t_k, H^n_k < 1$.

The 3×3 upper corner of the covariance matrix of the process $\mathbf{Y}^k_{(n)}$ for $a^t_k = a^n_k = 1$ is

$$\begin{aligned}
[(\Lambda^k)_{(n)}]_{3\times 3} =& \frac{1}{n^{2H^t_k}}[(\Gamma_{H^t_k})_{(n)}]_{3\times 3} + \frac{1}{n^{2H^n_k}}[(\Gamma_{H^n_k})_{(n)}]_{3\times 3} \\
=& \begin{pmatrix} f(H^t_k, H^n_k) & F(H^t_k, H^n_k) & g(H^t_k, H^n_k) \\ F(H^t_k, H^n_k) & f(H^t_k, H^n_k) & F(H^t_k, H^n_k) \\ g(H^t_k, H^n_k) & F(H^t_k, H^n_k) & f(H^t_k, H^n_k) \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
f(H^t_k, H^n_k) &= \frac{1}{n^{2H^t_k}} + \frac{1}{n^{2H^n_k}} \\
F(H^t_k, H^n_k) &= \frac{l(H^t_k)}{n^{2H^t_k}} + \frac{l(H^n_k)}{n^{2H^n_k}} \\
g(H^t_k, H^n_k) &= \frac{b(H^t_k)}{n^{2H^t_k}} + \frac{b(H^n_k)}{n^{2H^n_k}},
\end{aligned}$$

with $b(H)$ and $l(H)$ defined as in (2.25) and (2.24). Then the elements of the approximate inverse covariance matrix will be

$$c_1(H^t_k, H^n_k) = \frac{f(H^t_k, H^n_k) + g(H^t_k, H^n_k)}{f(H^t_k, H^n_k)[f(H^t_k, H^n_k) + g(H^t_k, H^n_k)] + 2F(H^t_k, H^n_k)}$$

and

$$c_2(H^t_k, H^n_k) = -\frac{F(H^t_k, H^n_k)}{f(H^t_k, H^n_k)[f(H^t_k, H^n_k) + g(H^t_k, H^n_k)] + 2F(H^t_k, H^n_k)}.$$

Since we are interested in the uniqueness of the solution of the Hurst parameters, we remark that also the elements c_1 and c_2 depend on the parameters H^t and H^n.

Using the same notation in section 2.3 for the stochastic processes $\mathbf{Y}^k_{(n)}$ with $k = A, B$, it follows

$$\mathbf{Y}^k_{(n)}(\Lambda^k)^{-1}_{(n)}(\mathbf{Y}^k_{(n)})^T = c_1(H^t_k, H^n_k)T^k_n + 2c_2(H^t_k, H^n_k)U^k_n.$$

Applying Corollary 2.34 to the sequences T^k_n and U^k_n in the limit $n \to \infty$, we get

$$\frac{T^k_n}{n} \sim f(H^t_k, H^n_k)$$

and

$$\frac{U^k_n}{n-1} \sim F(H^t_k, H^n_k).$$

Hence, the posterior distribution for the process $\mathbf{Y}^k_{(n)}$ given $H^t = H^t_k$ and $H^n = H^n_k$ equals

$$\begin{aligned} &D^{app}_{post}(\mathbf{Y}^k_{(n)}, H^t, H^n) = \beta(2\pi)^{-\frac{n}{2}} \left|(\Lambda^k)_{(n),app}\right|^{-\frac{1}{2}} \\ &\cdot \exp\big(-\frac{n}{2}(c_1(H^t_k, H^n_k)f(H^t_k, H^n_k) + 2c_2(H^t_k, H^n_k)F(H^t_k, H^n_k))\big). \end{aligned} \tag{3.15}$$

The problem of estimating the Hurst parameters H^t and H^n of the sum of two FBMs with amplitude parameters known leads the following theorem:

Theorem 3.16. *Consider the stochastic processes $\mathbf{Y}^A_{(n)}$ and $\mathbf{Y}^B_{(n)}$ defined in (3.14) with $\frac{1}{2} < H^t_k, H^n_k$ <1, where $k = A, B$. Then*

$$D^{app}_{post}(\mathbf{Y}^A_{(n)}, H^t, H^n) = D^{app}_{post}(\mathbf{Y}^B_{(n)}, H^t, H^n)$$

for $n \to \infty$ implies

$$H^n_A = H^n_B.$$

Proof. Starting from (3.15), we can easily verify that in the limit $n \to \infty$

$$D^{app}_{post}(\mathbf{Y}^A_{(n)}, H^t, H^n) = D^{app}_{post}(\mathbf{Y}^B_{(n)}, H^t, H^n)$$

if and only if

$$\begin{aligned} &\lim_{n\to\infty} \frac{\beta}{(2\pi)^{\frac{n}{2}}} \det\Lambda^{-\frac{1}{2}} \exp\big(-\frac{n}{2}(c_1(H^t, H^n)f(H^t_A, H^n_A) + 2c_2(H^t, H^n)F(H^t_A, H^n_A))\big) = \\ &\lim_{n\to\infty} \frac{\beta}{(2\pi)^{\frac{n}{2}}} \det\Lambda^{-\frac{1}{2}} \exp\big(-\frac{n}{2}(c_1(H^t, H^n)f(H^t_B, H^n_B) + 2c_2(H^t, H^n)F(H^t_B, H^n_B))\big) \end{aligned}$$

with $\det\Lambda = \left|(\Lambda^k)_{(n),app}\right|$. This is equivalent to

$$\begin{aligned} &\lim_{n\to\infty} \big[f(H^t, H^n) + g(H^t, H^n)\big]\big[f(H^t_A, H^n_A) - f(H^t_B, H^n_B)\big] = \\ &\lim_{n\to\infty} 2F(H^t, H^n)\big[F(H^t_A, H^n_A) - F(H^t_B, H^n_B)\big]. \end{aligned} \tag{3.17}$$

We can here use the same argument as in the proof of Theorem 3.11. Since we suppose that $H^t_A > H^n_A$ and $H^t_B > H^n_B$, the dominating terms in the limit as $n \to \infty$, will be of the order $1/n^{2(H^n_A+H^n_B)}$. Hence, considering only the terms of this order, Equation (3.17) can be written as

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^{2(H^n_A+H^n_B)}}\big(1 + b(H^n_A) - 1 - b(H^n_B)\big) = \\ &\lim_{n\to\infty} \frac{-2}{n^{2(H^n_A+H^n_B)}} l(H^n)\big(l(H^n_A) - l(H^n_B)\big) \end{aligned}$$

if and only if

$$l(H^n_A) = l(H^n_A).$$

This is satisfied if and only if $H^n_A = H^n_B$. ■

Our goal was to prove the uniqueness of the solution with respect to both Hurst parameters. We have only succeeded in proving the uniqueness with respect to the smaller of the Hurst parameters, H^n.

On the other hand, computer simulations were successful in some cases in retrieving both parameters quite accurately. We think that when studying the asymptotic behavior of the posterior distribution analytically, we lose some information present in the simulations.

Of course, the loss of information may be due to the fact that we consider the approximate inverse covariance matrix. It is possible that taking a band matrix with a larger band could lead to better results, but the calculations would get much more complicated.

This remains an open problem.

Chapter 4

Numerical simulations

Numerical simulations play a central role in this work. In fact, this work started with computer simulations. The theory, as presented in previous chapters, has been developed on the basis of the numerical results. In this chapter, we compare the analytical results with the obtained simulation results. As pointed out in previous chapters, the numerical simulations show more features than those we have succeeded in proving analytically. The simulations were done in MATLAB.

The first step in the simulations is to generate FBM realizations, given the Hurst parameter and the number of measurements. All realizations are performed for $t \in [0, 1]$.

In the next paragraphs, we show how the posterior distributions behave in the following cases:

1. Retrieving the amplitude parameter from a realization on FBM
2. Retrieving the Hurst parameter from a realization of FBM
3. Retrieving the underlying parameters from a measurement vector which is the sum of two realizations of FBM.

4.1 Retrieving the amplitude parameter from a realization of a FBM

To study the behavior of the posterior distribution when retrieving the amplitude parameter from a realization of FBM, we first generate a FBM realization Z_H with a given Hurst parameter H. Then we choose a value for the amplitude parameter $\hat{a}$ and calculate the measurement vector $\mathbf{X}$ as in Equation (2.2) and calculate the posterior distribution $D_{post}(\hat{a}, \mathbf{X})$ as shown in Equation (2.3).

The resulting posterior distribution is shown in Figure 1.2.

The use of different values for the Hurst parameter has no effect on the behavior of the posterior distribution. We have not found any relation between the Hurst parameter and the amplitude parameter that could influence the behavior of the posterior distribution of the amplitude.

On the other hand, the number of observations n in the measurement vector quite strongly affects the posterior distribution $D_{post}(\hat{a}, \mathbf{X})$.

To study what happens when the number of observations increases, we generated a realization of FBM with 2000 points ($t_0 = 0, ..., t_{2000} = 1$). From this realization, we created measurement vectors of different lengths by picking up observations at times t_{nh} with h fixed and $n = 0, ..., 2000/h$.

We calculated the resulting posterior distributions using the same realization, but doubling each time the number of observations.

Table 4.1 shows the numerical results obtained for the amplitude parameter in a FBM realization when the Hurst parameter equals 0.7, the amplitude $\hat{a}$ equals 50 and $\bar{a}$ is the amplitude estimate.

Table 4.1: Posterior distribution maxima as a function of n with $a \in (1, 800)$.

n	$\max(D_{post}(\hat{a}, \mathbf{X}))$	$\bar{a}$
26	0.10	33
51	0.08	47
101	0.13	43
201	0.15	51
401	0.25	49

It is easy to see that the posterior distribution becomes more peaked and the estimate of the amplitude parameter more reliable as the number of observations increases. A graphical representation of this behavior is shown in Figure 1.3.

4.2 Retrieving the Hurst parameter from a realization of FBM

To study the behavior of the posterior distribution when retrieving the Hurst parameter, we proceeded as with the amplitude parameter. We first generated a FBM realization $Z_{\hat{H}}$ with a given Hurst parameter $\hat{H}$. Then we chose the amplitude parameter a to make the measurement vector $\mathbf{Z} = \sqrt{a} Z_{\hat{H}}$ and calculated the posterior distribution $D_{post}(\hat{H}, \mathbf{Z})$ as shown in Equation (2.40).

As in the case of the amplitude parameter, we could not find any relation between the values of the Hurst parameter H and the amplitude parameter a that could influence the behavior of the posterior distribution $D_{post}(\hat{H}, \mathbf{Z})$.

On the other hand, the number of observations has a significant effect on the behavior of the posterior distribution with respect to $\hat{H}$.

In the simulations, we used the same measurements as with the amplitude parameter. This means that both the original vector (i.e. the one containing 2001 observations) and the shorter vectors are the same as used in the previous section.

Table 4.2: Posterior distribution maxima as a function of n with $H \in (0, 1)$.

n	$\max(D_{post}(\hat{H}, \mathbf{X}))$	$\bar{H}$
26	12.50	0.76
51	19.10	0.70
101	30.65	0.71
201	47.28	0.70
401	10^{156}	0.70

Table 4.2 shows the numerical results obtained for the Hurst parameter in a FBM realization when the amplitude equals 50, the Hurst parameter 0.7 and $\bar{H}$ is its estimate.

It is easy to see that the estimate becomes more precise and the posteriors distribution more peaked as the number of observations increases. A graphical representation of this behavior is given in Figure 2.2.

An interesting feature can be noted comparing Tables 4.1 and 4.2. First of all, we cannot say, just on the basis of the simulation results, that the posterior distribution for $\hat{a}$ tends to Dirac's delta distribution. We can only say that the estimate of the amplitude parameter becomes narrower when the number of observation increases.

On the other hand, the posterior distribution of $\hat{H}$ is more peaked and gets higher at a faster rate than the amplitude distribution. In particular, it seems that after a certain point, the posterior distribution really is like Dirac's delta.

Our goal has been to prove that the posterior distribution concentrates on the exact value of the estimated parameter. We have not analyzed the rate of convergence of the posterior distribution. However, it will be useful to keep in mind this behavior difference between the two posterior distributions in order to better understand the numerical results obtained from the simulations of the sum of two FBMs.

4.3 Retrieving the parameters from the sum of two FBM realizations

To retrieve the underlying parameters from a measurement vector which is the sum of two FBMs, we have to build the simulation through the following steps:

1. Generate two FBM realizations, X^t and X^n with given underlying parameters a^t, H^t, a^n, H^n.
2. Make $\mathbf{Z} = X^t + X^n$ the measurement vector.
3. Calculate the posterior distribution according to

$$D_{post}(H^t, H^n, a^t, a^n, \mathbf{Z}) = (2\pi)^{-n/2} \left|\Sigma_n + \Sigma_t\right|^{-1/2} \exp(-\frac{1}{2}\mathbf{Z}^T(\Sigma_n + \Sigma_t)^{-1}\mathbf{Z}),$$

   where Σ_n and Σ_t are the covariance matrices of the stochastic processes X_n and X_t, respectively.
4. Calculate the marginal distributions of each of the unknown parameters and plot them.

The idea in taking the sum of two FBMs is that we also consider the error to be a FBM realization. Usually, in this kind of simulations, once the realization is given, the error is added in order to model the measurement vector. In practice, this is usually done by generating a vector of random variables with Gaussian distributions and adding it to the realization vector (see e.g. [5]).

Since we suppose here that also the error is a FBM realization in the vector Z, both the trend and the error are modelled. The noise vector is characterized by an amplitude smaller than that of the trend, and a smaller Hurst parameter.

As a result of each simulation, we obtain four different posterior (marginal) distributions: $D_{post}(H^t, a^t, \mathbf{Z})$, $D_{post}(H^n, a^n, \mathbf{Z})$, $D_{post}(H^t, H^n, \mathbf{Z})$ and $D_{post}(a^t, a^n, \mathbf{Z})$.

The first observation from the simulations is that $D_{post}(H^t, a^t, \mathbf{Z}) = D_{post}(H^n, a^n, \mathbf{Z})$ and, in particular, that we get an estimator of the *smallest* Hurst parameter, H^t or H^n. This result is fully supported by the analytical results, where we stated that in the case of the sum of two FBMs, only the smaller Hurst parameter can be uniquely estimated.

To study the behavior of the posterior distributions from a sum of two FBMs, we simulated the measurement vector with different values of the underlying parameters. We show the most relevant simulation results in a few figures.

First, the case where $H^n = H^t = 0.3$, $a^n = 100$ and $a^t = 500$.

As seen in Figure 4.1, the posterior distribution $D_{post}(H^n, a^n, \mathbf{Z})$ attains its maximum for $H^n = 0.28$ and a^n between 500 and 600. $D_{post}(H^n, a^n, \mathbf{Z})$ attains a local maximum for $H^n = 0.22$ and a^n around 350. It is obvious that $D_{post}(H^n, a^n, \mathbf{Z})$ is quite peaked with respect to H^n, but not with respect to a^n.

Furthermore, the posterior distribution $D_{post}(a^t, a^n, \mathbf{Z})$ is quite flat and gives for all values of the amplitude parameters a^n and a^t a probability different from zero. The posterior distribution $D_{post}(a^t, a^n, \mathbf{Z})$ is symmetric, since it is not possible to choose between the values of a^n and a^t. The posterior distribution $D_{post}(a^t, a^n, \mathbf{Z})$ attains its maximum for $(a^t, a^n) = (600, 1)$. In particular, all the values on the line $a^t + a^n = 600$ have a high probability.

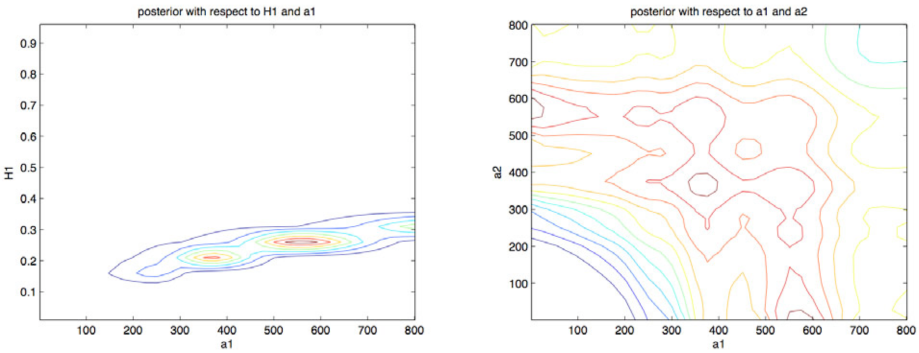

Figure 4.1: Posterior distributions for the sum of two FBMs with $H^t = H^n = 0.3, a^t = 500, a^n = 100$.

This numerical result supports the analytical result shown in section 3.1. We can conclude that, in the case of a measurement vector which is the sum of two FBMs with equal Hurst parameters, the amplitude parameter estimates obtained by the statistical inversion method are not reliable.

The next case is $H^n = 0.1$, $H^t = 0.9$ and $a^t = a^n = 200$ to check how having the same value of the amplitude parameter affects the posterior (marginal) distributions.

Figure 4.2 shows both $D_{post}(H^t, H^n, \mathbf{Z})$ and $D_{post}(a^t, a^n, \mathbf{Z})$. As with the amplitude parameters, $D_{post}(H^t, H^n, \mathbf{Z})$ is always symmetric. In particular, $D_{post}(H^t, H^n, \mathbf{Z})$ attains its maximum for $(H^t, H^n) = (0.98, 0.13)$.

However, $D_{post}(a^t, a^n, \mathbf{Z})$ attains its maximum for $(a^t, a^n) = (165, 165)$.

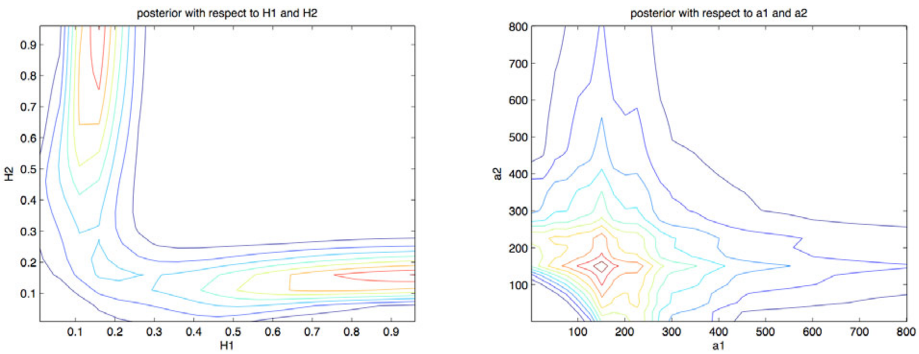

Figure 4.2: Posterior distributions for the sum of two FBMs with $H^t = 0.9, H^n = 0.1, a^t = 200, a^n = 200$.

We deduce that the underlying parameter estimates of the sum of two FBMs with the same amplitude are quite reliable.

All the analytic results on retrieving the Hurst parameters are restricted to the case of H^t and $H^n > \frac{1}{2}$ for technical reasons.

The numerical results for the sum of FBMs with $H^t = 0.3$, $H^n = 0.1$, $a^t = 500$ and $a^n = 100$ are shown in Figure 4.3. It is possible to see that $D_{post}(H^t, H^n, \mathbf{Z})$ attains its maximum for $(H^t, H^n) = (0.28, 0.22)$, but this maximum is not as as sharp as in the previous simulation. This indicates uncertainty on the value of the estimate. Moreover, there is also a local maximum for $(H^t, H^n) = (0.92, 0.20)$ where the posterior distribution attains a value only slightly smaller than for $(H^t, H^n) = (0.28, 0.22)$.

On the other hand, $D_{post}(a^t, a^n, \mathbf{Z})$ attains its maximum for $(a^t, a^n) = (600, 1)$. However, the posterior distribution is quite flat and small in value. This means that the information obtained from $D_{post}(a^t, a^n, \mathbf{Z})$ about the most probable value of (a^t, a^n) is quite poor.

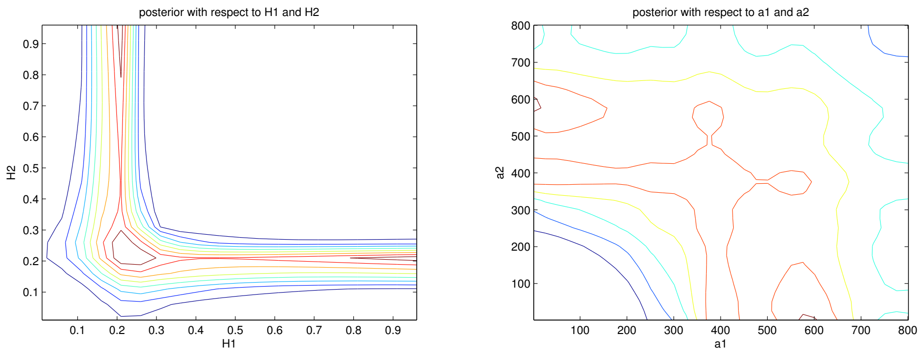

Figure 4.3: Posterior distributions for the sum of two FBMs with $H^t = 0.3, H^n = 0.1, a^t = 500, a^n = 100$.

It is obvious that the parameter estimates are not reliable in the case of a measurement vector which is the sum of two FBMs with both Hurst parameters less than $\frac{1}{2}$.

If both Hurst parameters are greater than $\frac{1}{2}$, we have the same result. Figure 4.4 shows that $D_{post}(H^t, H^n, \mathbf{Z})$ attains its maximum for $(H^t, H^n) = (0.82, 0.75)$, giving a hint that both Hurt parameters are indeed greater than $\frac{1}{2}$. The posterior distribution $D_{post}(H^t, H^n, \mathbf{Z})$ is much sharper than in the previous case and attains values different from zero in smaller area than in Figure 4.3.

The posterior distribution $D_{post}(a^t, a^n, \mathbf{Z})$ attains its maximum for $(a^t, a^n) = (800, 1)$. Also in this case, the posterior distribution is quite flat and small in value. Moreover, there is an area of (H^t, H^n) greater than 700 where the probability distribution $D_{post}(a^t, a^n, \mathbf{Z})$ attains quite a high value. This indicates again that there is uncertainty on the value of the amplitude parameter estimates.

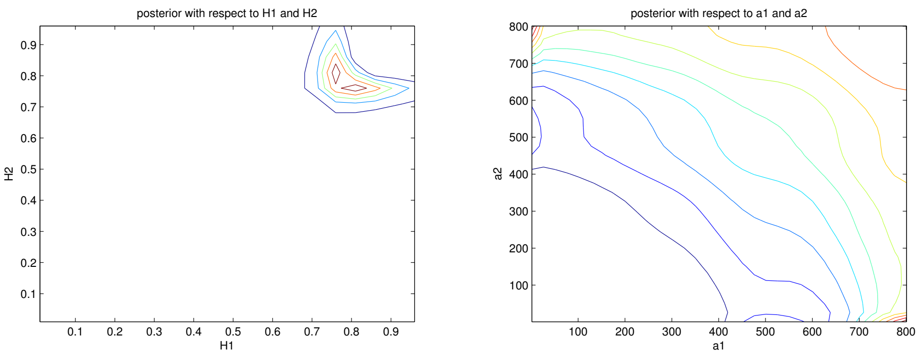

Figure 4.4: Posterior distributions for the sum of two FBMs with $H^t = 0.6, H^n = 0.8, a^t = 500, a^n = 100$.

Concluding, we observe that the estimates for the underlying parameters are not reliable in the case of the sum of two FBMs with both Hurst parameters greater than $\frac{1}{2}$.

This suggests that the FBM realizations from which we generate the measurement vector as their sum should be different enough so that they can be better recognized in the posterior distribution.

For this reason, we show the numerical results obtained in the case of the sum of two FBMs with $H^t = 0.9$ and $H^n = 0.1$. Figure 4.5 shows the posterior distributions for the case where a^t and a^n are the same as in the two previous simulations, i.e. $a^t = 500$ and $a^n = 100$.

$D_{post}(H^t, H^n, \mathbf{Z})$ attains its maximum for $(H^t, H^n) = (0.93, 0.15)$. This is the first time in our simulations that the estimates are close to the correct values of the Hurst parameters. Although the posterior distribution is not as peaked as in Figure 4.4, the maximum value is greater. Comparing the two posterior distributions $D_{post}(H^t, H^n, \mathbf{Z})$ shown in Figures 4.4 and 4.5, also the area where they attain their maxima is smaller in latter case, indicating that there is a smaller degree of uncertainty in Figure 4.5.

On the other hand, the posterior distribution $D_{post}(a^t, a^n, \mathbf{Z})$ attains its maximum for $(a^t, a^n) = (115, 100)$. In this case, the posterior distribution is not so small as in the previous two cases. However, the amplitude parameter estimates are not at all what we would expect.

From these numerical results we can deduce that the Hurst parameters obtained in the case of the sum of two FBMs with $H^t = 0.9$, $H^n = 0.1$, $a^t = 500$ and $a^n = 100$, are reliable. However, the amplitude parameter estimates are not reliable.

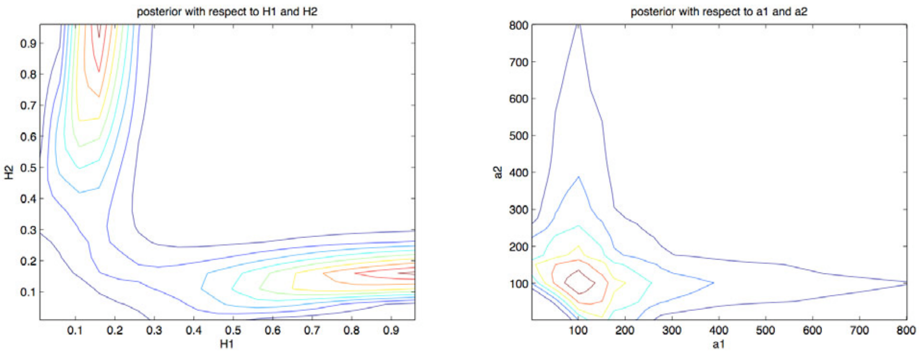

Figure 4.5: Posterior distributions for the sum of two FBMs with $H^t = 0.9, H^n = 0.1, a^t = 500, a^n = 100$.

Next, we study how different values of the amplitude parameters affect the posterior distribution.

We have simulated two measurement vectors which are the sums of two FBMs where $H^t = 0.9$ and $H^n = 0.1$ as in the previous simulation, but the ratio between the amplitude parameters changes. Figure 4.6 shows the posterior (marginal) distributions with $a^t = 500$ and a^n equal to 10 and 1, respectively.

The shape of $D_{post}(H^t, H^n, \mathbf{Z})$ does not differ so much in the two cases. However, the estimates for the Hurt parameters become more precise in the second case with $(0.98, 0.22)$ versus the first one $(0.87, 0.12)$.

It seems that the results obtained with $a^n = 10$ are worse than those obtained with $a^n = 100$. It is not so, if we also compare the posterior distributions $D_{post}(a^t, a^n, \mathbf{Z})$.

We see that the posterior distributions $D_{post}(a^t, a^n, \mathbf{Z})$ peak more quickly and give better and better estimates both for a^t and a^n, also in comparison with Figure 4.5.

Previously, there was only one FBM realization, whereas here, for each calculation, two new FBMs are generated with different parameters. This implies that the estimates cannot always be satisfactory.

On the other hand, by studying the shapes of the posterior distributions and comparing all three posterior distributions, we can get much more information than only by taking the maximum values of the posterior distributions.

From the numerical results we deduce that the estimates for the underlying parameters become more reliable as the ratio a^t/a^n increases.

All simulations in the estimation of the underlying parameters of the sum of two FBM realizations were performed with measurement vectors containing a constant number, 500, of observations, so that the results obtained are consistent.

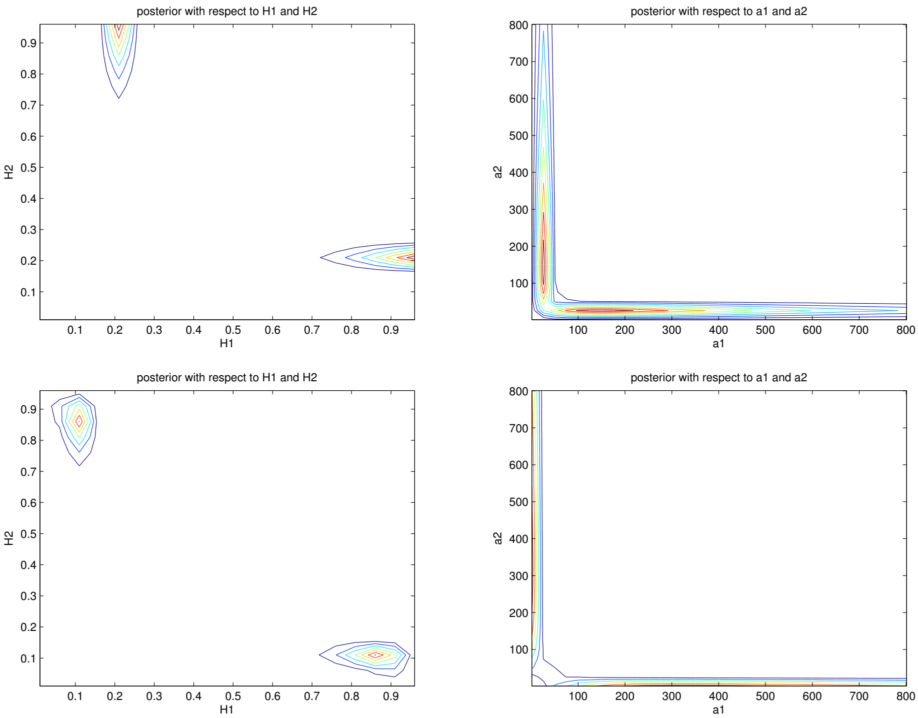

Figure 4.6: Posterior distributions for the sum of two FBMs with $H^t = 0.9, H^n = 0.1, a^n = 500$ and a^t equal to 10 and 1, respectively.

It is clear that with a larger number of observations we could obtain better estimates for the underlying parameters.

4.4 Reconstruction

Once the underlying parameters have been estimated, we try to reconstruct the components of the measurement vector. The goal is to obtain the trend vector as accurately as possible.

This is done in the following way:

1. Given the estimated parameters H^n and a^n generate a realization of FBM.
2. Calculate $X^t = Z - X^n$.

Figure 4.7 shows the reconstructions of the trend and noise vectors from the sum of two FBMs with $H^t = 0.9$ and $H^n = 0.1$ and with $a^t = 500$ and $a^n = 1$.

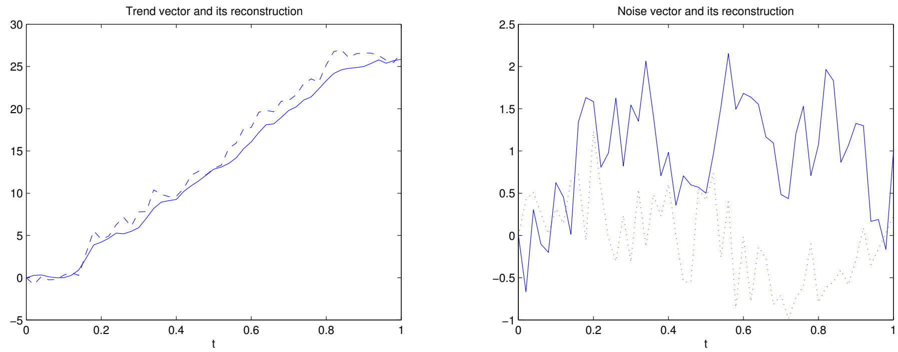

Figure 4.7: Reconstruction of the trend and noise vectors given the sum of two FBMs with $H^t = 0.9$ and $H^n = 0.1$ with $a^t = 500$ and $a^n = 1$.

As shown in the previous section, the statistical inversion method gives quite accurate results in estimating the underlying parameters in cases such this one. From Figure 4.7, we can observe that also the reconstruction is quite successful.

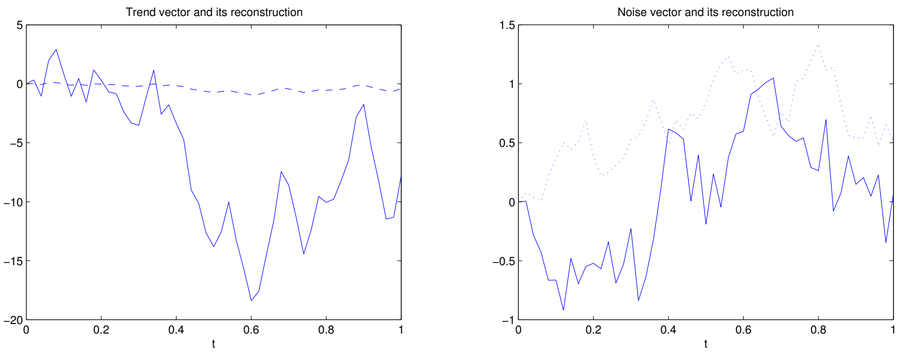

Figure 4.8: Reconstruction of the trend and noise given the sum of two FBMs with $H^t = 0.6$ and $H^n = 0.3$ with $a^t = 500$ and $a^n = 1$.

Figures 4.8 show the reconstructions of the trend and noise vectors when the measurements vector is the sum of two FBMs with $H^t = 0.6$ and $H^n = 0.3$ and with $a^t = 500$ and $a^n = 1$.

We can see that when Hurst parameters are not very different, the reconstruction is quite strongly affected. This is obviously a direct consequence of the fact that the statistical inversion method does not give a good estimator of the Hurst parameters. Figure 4.8 shows that in this case, the reconstruction is no more reliable.

In the last example, the Hurst parameters are the same as in Figure 4.7 and the amplitude parameters are closer to each other. Figures 4.9 shows the reconstructions

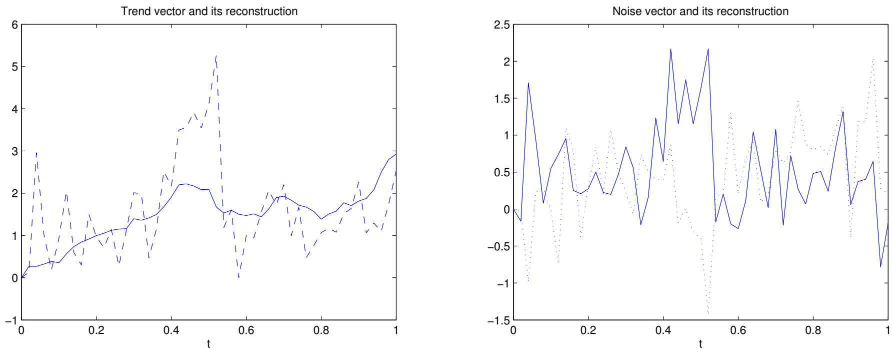

Figure 4.9: Reconstruction of the trend and noise vectors given the sum of two FBMs with $H^t = 0.9$ and $H^n = 0.1$ and with $a^t = 50$ and $a^n = 1$.

of the trend and noise vectors of the sum of two FBMs with $H^t = 0.9$ and $H^n = 0.1$ with $a^t = 50$ and $a^n = 1$.

Also in this case, one can see that when the amplitude parameters are not that much different from each other, the reconstruction is affected. The effect is not as strong as in Figure 4.8, because the estimate of the Hurst parameter H^n is equal to the correct value. Even in this case, the trend reconstruction is not reliable.

Chapter 5

Conclusions

Here we summarize the simulation results.

The first result on one FBM realization is that, both when retrieving the amplitude parameter or the Hurst parameter, the estimates become more reliable with increasing the number of observations. The rate of convergence of the estimates to the correct values of the underlying parameters is faster for the Hurst parameter than for the amplitude parameter.

For the sum of two FBMs, there are many results. Table 5.1 shows a summary of how different changes in the measurement vector affect the posterior distributions $D_{post}(H^t, H^n, \mathbf{Z})$ and $D_{post}(a^t, a^n, \mathbf{Z})$. The last two columns are self-explanatory.

Table 5.1: Posterior distribution maxima as a function of n for some combinations of H^t, H^n and a^t, a^n.

Measurement vector	$D_{post}(H^t, H^n, \mathbf{Z})$	$D_{post}(a^t, a^n, \mathbf{Z})$
$H^t = H^n$	👎	—
$a^t = a^n$	—	—
$H^t, H^n < \frac{1}{2}$	👎	👎
$H^t, H^n > \frac{1}{2}$	👎	👎
$H^t = 0.9$ $H^n = 0.1$ $a^t = 500$ $a^n = 100$	👎	👍
$H^t = 0.9$ $H^n = 0.1$ $a^t = 500$ $a^n = 10$	👍	👍
$H^t = 0.9$ $H^n = 0.1$ $a^t = 500$ $a^n = 1$	👍	👍
Increasing the number of observations	👍	👍

It can be concluded that the best results are obtained for H^t near one, H^n nearer zero, a^t large with respect to a^n and a^n near one.

These conditions seem restrictive, but in reality, they are not. If we consider one FBM realization to represent the trend and the other one to represent the noise, the trend is usually quite smooth, so it is tempting to conjecture that the FBM which represents the trend would a Hurst parameter near one.

On the other hand, in physics, in many cases the noise is about a few per cent, so it is consistent to think that the amplitude parameter of the noise FBM should be much smaller than that of the trend.

We are able to model physical measurements using a *single* stochastic process and simplify the calculation procedure.

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